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APPROXIMATION FOR DOMINATING SET PROBLEM WITH MEASURE FUNCTIONS*

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Abstract. In this paper, we study the Dominating Set problem with measure functions, which is extended from the general Dominating Set problem. We study the correspondnig problems on complexity, approximation and inapproximability for Dominating Set problem with measure functions. In addition, we extend our results to the weighted graphs.

Keywords: Dominating set, complexity, approximation, inapproximability

1 INTRODUCTION

Approximation algorithms [12, 22], which are used to solve optimization problems in polynomial time with produced approximate solution being guaranteed to be close to the optimal solution, is a major research area in theoretical computer science.

For minimization problems, an algorithm achieves approximation ratio $\delta \geq 1$ if for any instance of the problem it produces a solution of value at most $\delta \cdot OPT$, where OPT is the optimal solution of the instance. We may define the approximation ratio for maximization problem in the same way. Specifically, the study of the approximation ratio contains two directions: upper bound, in which we need to design algorithm that achieves ratio δ, and lower bound, in which we should prove

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that no approximation algorithms can achieve ratio δ for the given optimization problem, unless $P = NP$.

In this paper, we extend the Dominating Set problem [11] to the one with measure functions, i.e., Measure Dominating Set. The measure function is any positive function $f(\cdot)$ defined on the number of vertices of the given graph. For any undirected graph $G = (V, E)$, the Measure Dominating Set problem asks for a minimal set $D \subseteq V$ such that for any vertex $v \notin D$, there exists $u \in D$ satisfying the length of the shortest path between u and v is at most $f(|V|)$. Note that in the traditional Dominating Set problem, we only require $(u, v) \in E$, that is, $f(|V|) \equiv 1$. In addition, we study this problem further on the graphs with weighted vertices and edges, i.e., Weighted-Measured Dominating Set. Similar works on the extensions of Dominating Set are referred to, e.g., in [6, 8].

Our contributions are the following threefolds for the both Measured and Weighted Dominating Set problem: (i) We show that both problems are *NP*-hard for measure function $f(n) = n^\varepsilon$, where $0 < \varepsilon < 1$ is any real. Note that Chang and Nemhauster [6] showed that for any fixed constant distance, i.e., $f(n) = c > 0$, the problem is *NP*-hard. (ii) We study approximation algorithms for the two problems. Specifically, for Measured Dominating Set, we give a deterministic approximation scheme with ratio $O(\log n)$; for the Weighted problem, we first show a deterministic algorithm with approximation ratio $O(\log n)$, then we give a probabilistic approximation algorithm with ratio $O(\log \Delta)$ by randomized rounding, where Δ is the maximum cardinality of the sets of vertices that can be dominated by any single vertex. (iii) We consider the inapproximability of the problems and demonstrate that both lower bounds are $\Omega(\log n)$.

In Section 2, we review the general Dominating Set problem and related works for Set Cover (hence, for Dominating Set) problem. In Section 3, we formally define the Measured Dominating Set problem and show our results on complexity, approximation and inapproximability. We extend the model to weighted case and discuss related problems in Section 4. We conclude our work in Section 5.

2 PRELIMINARIES

Given undirected connected graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ and E is the set of edges, the Dominating Set problem is defined as follows.

Definition 2.1 (Dominating Set)**.** For any graph $G = (V, E)$, we are asked to find a dominating set $D \subseteq V$ with minimal cardinality, termed as optimal dominating set, such that for any $v_j \in V - D$, there exists $v_i \in D$ satisfying $(v_i, v_j) \in E$.

Dominating Set is a classic *NP*-hard problem, whose *NP*-hardness is reduced from Set Cover problem [10], and has been discussed extensively from the point of view of approximation. Note that all approximation and inapproximability results of Set Cover also apply to Dominating Set similarly; hence, in the sequel we briefly review the related works on Set Cover problem.

For approximation algorithm, both greedy and linear program relaxation approach work for the approximation of Set Cover with the same approximation ratio $H(n) = \sum_{i=1}^{n} \frac{1}{n} \leq \ln n + 1$ [13, 14, 20, 21]. Chvátal [7] extended the result to the weighted case. Therefore, these solutions generate $O(\log n)$ approximation-ratio algorithms for Dominating Set problem.

For inapproximability result, first, Arora et al. [2] showed that for some $\varepsilon > 0$, it is *NP*-hard to approximate Set Cover problem within ratio $1 + \varepsilon$, on the basis of probabilistic checkable proof [2, 3] and MAX-*SNP* [17]. Secondly, Lund and Yannakakis [15] proved that Set Cover cannot be approximated within ratio $c \log n$ for any $c < \frac{1}{4}$, unless $NP \subset TIME(n^{O(\mathrm{poly} \log n)})$. After that, Bellare et al. [5] showed that unless $P = NP$, Set Cover cannot be approximated within any constant ratio; Raz [18] proved the inapproximability ratio $c \log n$ for any $c < \frac{1}{4}$, unless $NP \subset TIME(n^{O(\log \log n)})$; and then Naor et al. [16] improved the coefficient $\frac{1}{4}$ to $\frac{1}{2}$ under the same assumption. Finally, Feige [9] closed the gap between approximation and inapproximability, i.e., for any $\varepsilon > 0$, Set Cover cannot be approximated within ratio $(1 - \varepsilon) \ln n$, unless $NP \subset TIME(n^{O(\mathrm{poly} \log n)})$; further, Arora, Sudan [4], and Raz, Safra [19] independently showed that approximating Set Cover within ratio $\Omega(\log n)$ is *NP*-hard, which thoroughly completes the work on the inapproximability of Set Cover (i.e., Dominating Set).

3 DOMINATING SET WITH MEASURE FUNCTIONS

Now we consider Dominating Set problem with *measure function* $f(n)$, where n is the number of vertices of the given graph. That is, for any $v_i \in V$ we say v_i *dominates* $v_j \in V$ if and only if there exists a path, with length not more than $f(n)$, from v_i to v_j, and vice versa. Let $LSP(v_i, v_j)$ denote the length of the shortest path between v_i and v_j. Formally,

Definition 3.1 (Measured Dominating Set)**.** For any given $G = (V, E)$ and measure function $f(n)$, where $n = |V|$, we are asked to find a dominating set $D \subseteq V$ with minimal cardinality, termed as optimal measured dominating set, such that for any $v_j \in V - D$, there exists $v_i \in D$ satisfying $LSP(v_i, v_j) \leq f(n)$.

3.1 *NP*-Hardness

Note that the general Dominating Set problem we discussed in the above section is a special case of $f(n) \equiv 1$. Thus, intuitively, Measured Dominating Set is at least as hard as the former one. Formally,

Theorem 3.1. For any real ε, $0 < \varepsilon < 1$, Measured Dominating Set problem is *NP*-hard for the measure function n^{ε}, where n is the number of vertices.

Proof. We reduce from the general Dominating Set problem (i.e., $f = 1$) on the graph $G = (V, E)$. Let δ be an integer that satisfies

$$1 \leq \delta \leq (\delta n)^{\varepsilon} < \delta + 1,$$

where $n = |V|$. It is easy to see that such an integer always exists, and upper bounded by $n^{\frac{\varepsilon}{1-\varepsilon}}$, which implies that the integer δ we need is bounded by the polynomial of n.

We construct the graph $G' = (V', E')$ with measure function n^ε as follows. For every vertex $v_i \in V$, we add other $\delta - 1$ vertices $v_i^1, \ldots, v_i^{\delta-1}$ with edges among them sequently. That is, define

$$\begin{aligned} V' &= V \cup \left\{ v_i^1, \ldots, v_i^{\delta-1} \mid i = 1, \ldots, n \right\}, \\ E' &= E \cup \left\{ (v_i, v_i^1), (v_i^1, v_i^2), \ldots, (v_i^{\delta-2}, v_i^{\delta-1}) \mid i = 1, \ldots, n \right\}. \end{aligned}$$

Note that in G', $|V'| = \delta n$, hence the measure function of G' is $f' = (\delta n)^\varepsilon$.

For any dominating set D of G with measure function f, it is easy to see that D is still a dominating set of G' with measure function f'. This is because for any $v_i^j \in V'$, $1 \le j \le \delta - 1$, there must exist $v_l \in D$ that dominates v_i with length one, i.e., $(v_l, v_i) \in E$. And then, $LSP(v_l, v_i^j) \le 1 + j \le \delta \le f'$ implies that v_l dominates v_i^j with length f'.

Conversely, for any dominating set D' of G' with measure function f', it is easy to see that for all $1 \le i \le n$ we must have $\left| D' \cap \{v_i, v_i^1, \ldots, v_i^{\delta-1}\} \right| \le 1$. Therefore, for every $v_i^j \in D'$, $1 \le j \le \delta - 1$ we may replace v_i^j with v_i in D', which defines a new set D and maintains the property of domination. Thus, we have $|D| = |D'|$ and both of them are dominating sets of G' with measure function f'. In addition, we know that D is also a dominating set of G with measure function f. This is because $f' < \delta + 1$, and hence, for any $v_j \notin D$, there must exist $v_i \in D$ such that $(v_i, v_j) \in E$. Hence the theorem follows. □

Some comments are in place. Note that if $f(n) \ge n$, any single vertex serves as an optimal measured dominating set. Specifically, when $f(n) < n$ and $f(n) = \Theta(n)$, we have the following lemma.

Lemma 3.1. For any graph $G = (V, E)$, and measure function $f(n)$, where $n = |V|$, if $f(n) < n$ and $f(n) = \Theta(n)$, then Measured Dominating Set is polynomial-time solvable.

Proof. It's easy to see that there exist constant integers c and N such that for all $n \ge N$ we have $c \ge \frac{n}{f(n)}$. In the following we show that the size of the optimal solution is at most c, and then we only need to enumerate all possible c'-collections of vertices to find the optimal measured dominating set, for $c' = 1, \ldots, c$.

First, we compute the spanning tree T of G. Trivially, any measured dominating set of T is a feasible solution of the original graph G. Therefore, if we find a solution of T with size at most c, it ensures that the size of the optimal solution of G is at most c. Hence we finish the whole proof of the lemma.

We denote the generated dominating set by D (initially $D = \emptyset$). For any $v_i, v_j \in V$, let $l(v_i, v_j)$ be the length of the shortest path between v_i and v_j in T. The algorithm to find a measured dominating set of T is as follows:

1. For an arbitrary $v_i \in V$, we regard v_i as the root of T and search T by BFS. Let $LEAF$ be the collection of all leaves of T.
2. If the height of T is no more than $f(n)$, then $D \leftarrow D \cup \{v_i\}$, and return D.
3. Otherwise, let $v_k = \arg\max_{v_j \in LEAF} l(v_i, v_j)$. Note that $l(v_i, v_k) > f(n)$, which implies that there exists a vertex v_j contained in the path between v_i and v_k such that $l(v_j, v_k) = f(n)$. Let $D \leftarrow D \cup \{v_j\}$, and delete the subtree rooted at v_j.
4. Goto step 2.

Trivially, D is a feasible solution of T. In each iteration, we must delete at least $f(n)$ vertices. Therefore, the algorithm runs at most $c \geq \frac{n}{f(n)}$ iterations and there are at most c vertices added into D. □

In the rest of the paper, we only consider the case of $f(n) = n^{\varepsilon}$, $0 < \varepsilon < 1$, and all the results are referred to this specific measure function.

3.2 Approximation Algorithm

Similar as the general Dominating Set, we now consider approximation algorithm for the Measured Dominating Set problem.

For any graph $G = (V, E)$, we consider the adjacent matrix M to the power $f(n)$. Define graph $G' = (V, E')$, where the set of edges E' is determined according to the Boolean matrix M', i.e., M' is the adjacent matrix of G'.

Lemma 3.2. D' is an optimal measured dominating set of G' with measure function $f' = 1$ if and only if D' is an optimal measured dominating set of G with measure function $f(n)$.

Proof. We only prove the necessary condition, the sufficient condition is similar. First, it is easy to see that D' does be a dominating set of G. Assume, on the contrary, that there exists a dominating set D of G with measure function $f(n)$ such that $|D| < |D'|$. That is, for any $u \in V - D$, there exists $v \in D$ such that $LSP(u, v) \leq f(n)$, which implies that $(u, v) \in E'$. Therefore, D is also a dominating set of G' with measure function f'. A contradiction. □

Based on the above lemma, we get the following approximation scheme for Measured Dominating Set on the basis of approximation algorithm for the general Dominating Set problem.

Approximation Scheme for Measured Dominating Set:

Input: graph $G = (V, E)$ and measure function $f(n)$.

Algorithm:

1. compute adjacent matrix M of G and boolean matrix $M' = M^{f(n)}$.
2. construct graph $G' = (V, E')$, where M' is adjacent matrix of G'.
3. compute the dominating set D' of G' with measure function $f' = 1$ by approximation algorithm for the traditional Dominating Set.

Output: dominating set D' of G with measure function $f(n)$.

From Lemma 3.2, we get the following result.

Theorem 3.2. If the algorithm for the general Dominating Set used above has approximation ratio δ, then the above scheme for Measured Dominating Set problem has approximation ratio δ.

Corollary 3.1. There exists an approximation algorithm for Measured Dominating Set with approximation ratio $O(\log n)$.

3.3 Inapproximability

Note that in the proof of Theorem 3.1, we show the *NP*-hardness of Measured Dominating Set by reducing from the general Dominating Set problem. Furthermore, observe that the reduction we constructed has the property that the optimal solutions of both instances share the same cardinality. Therefore, we have showed that for any instance I of Dominating Set, the following gap-preserving reduction holds:

$$\begin{aligned} OPT(I) \leq c \quad &\Longrightarrow \quad OPT(\tau(I)) \leq c \\ OPT(I) > c\rho \quad &\Longrightarrow \quad OPT(\tau(I)) > c\rho \end{aligned}$$

where τ is the reduction we constructed in the proof of Theorem 3.1, c and $\rho \geq 1$ are functions of the instance size $|I|$. Due to the known inapproximability results for Set Cover (i.e., Dominating Set) demonstrated in Section 2, we have the following theorem.

Theorem 3.3. Measured Dominating Set problem cannot be approximated within ratio $\Omega(\log n)$ unless $P = NP$.

From Corollary 3.1 and the above theorem, we know that the bound for Measured Dominating Set problem is tight, i.e., $\Theta(\log n)$.

4 MEASURED DOMINATING SET FOR WEIGHTED GRAPHS

In this section, we study Measured Dominating Set problem on the graph with both non-negative weighted vertices (with weight function w) and edges (with weight function w'). In this case, the weighted-shortest path between v_i and v_j, denoted as $WSP(v_i, v_j)$, is the minimum sum of weights of edges on all such paths. That is,

$$WSP(v_i, v_j) = \min \left\{ \sum_{e_l \in P} w'(e_l) \mid P \text{ is a path between } v_i \text{ and } v_j \right\}.$$

Similarly, we say v_i *dominates* v_j if and only if $WSP(v_i, v_j) \leq f(n)$, and vice versa.

Definition 4.1 (Weighted-Measured Dominating Set)**.** For any given graph $G = (V, E)$ with weighted vertices and edges and measure function $f(n)$, where $n = |V|$, we are asked to find a dominating set $D \subseteq V$ with minimal (vertices) weights, termed as optimal weighted-measured dominating set, such that for any $v_j \in V - D$ there exists $v_i \in D$ satisfying $WSP(v_i, v_j) \leq f(n)$.

Note that in the above definition the criteria of the dominating set refer to vertices weights, whereas that two vertices are dominated each other refers to weights of edges.

Observe that Measured Dominating Set is a special case of the weighted model when all weights of vertices and edges are equal to one. Therefore, we have the following conclusion.

Theorem 4.1. Weighted-Measured Dominating Set problem is *NP*-hard.

In addition, the conclusion of Theorem 3.3 also works here, that is,

Theorem 4.2. Approximating Weighted-Measured Dominating Set problem within ratio $\Omega(\log n)$ is *NP*-hard.

Now we are searching for the approximation schemes for Weighted-Measured Dominating Set problem.

4.1 Greedy Algorithm

Similar as most other classic optimization problems, we first consider to apply the greedy approach.

For any given instance of Weighted-Measured Dominating Set problem, let OPT be the size of the optimal solution. Define $S_i = \{v_j \mid WSP(v_i, v_j) \leq f(n)\}$, and $\Delta = \max |S_i|$. Note that for all $1 \leq i \leq n$, $v_i \in S_i$. Let D be the dominating set generated by the algorithm, and $\overline{D} = V - D$. Let U^l be the set of vertices not dominated so far at the beginning of iteration l. The greedy algorithm works as follows.

Greedy Algorithm for Weighted-Measured Dominating Set:

1. $D \leftarrow \emptyset$, $U^1 \leftarrow V$, $l \leftarrow 1$.
2. $S_i^l \leftarrow S_i$, for $1 \leq i \leq n$.
3. While $U^l \neq \emptyset$ do
   (a) $v_j = \arg\min_{v_i \in \overline{D}} \frac{w(v_i)}{|S_i^l|}$.
   (b) $D \leftarrow D \cup \{v_j\}$.
   (c) $U^{l+1} \leftarrow U^l - S_j^l$.
   (d) $g(v_i) = w(v_j)/|S_j^l|$, for $v_i \in S_j^l$.
   (e) $S_i^{l+1} \leftarrow S_i^l - S_j^l$, for $v_i \in \overline{D}$.
   (f) $l \leftarrow l + 1$.
4. Output dominating set D with measure function $f(n)$.

Theorem 4.3. The above greedy algorithm produces an H_n ratio approximation algorithm for Weighted-Measured Dominating Set problem, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Proof. Assume without loss of generality that $v_1, \ldots, v_n$ is the order in which these vertices are removed (i.e., dominated) by step (iii) of the algorithm, ties are broken arbitrarily. It is easy to see that $g(v_1) \leq g(v_2) \leq \cdots \leq g(v_n)$. In addition, from the definition of (iv), we know that

$$\sum_{i=1}^{n} g(v_i) = \sum_{v_i \in D} w(v_i). \tag{1}$$

For any vertex v_i, assume v_i is dominated at iteration l, i.e., $v_i \in U^l$ and $v_i \notin U^{l+1}$. Hence, we know that $|U^l| \geq n - i + 1$. Therefore, from the minimization of (i), we have

$$g(v_i) \leq \frac{OPT}{|U^l|} \leq \frac{OPT}{n - i + 1}, \tag{2}$$

where the first inequality is due to at the beginning of iteration l, the leftover sets of the optimal solution (contained in $\overline{D}$) can dominate U^l at a cost of at most OPT. Combining (1) and (2), we have

$$\sum_{v_i \in D} w(v_i) = \sum_{i=1}^{n} g(v_i) \leq \sum_{i=1}^{n} \frac{OPT}{n - i + 1} = H_n \cdot OPT.$$

Hence the theorem follows. □

It is well known that $H_n \leq \log n + 1$, therefore the approximation ratio produced by greedy algorithm is upper bounded by $O(\log n)$. In addition, we stress that the bound H_n is tight. The following example demonstrates this point clearly.

Example 4.1. Consider the graph $G = (V, E)$ with measure function $f(n+1) = 1$, where $V = \{v_1, \ldots, v_n, v_{n+1}\}$ (with weights $w(v_i) = \frac{1}{i}$, for $1 \leq i \leq n$, and $w(v_{n+1}) = 1 + \varepsilon$ for a sufficiently small real $\varepsilon > 0$), and $E = \{(v_1, v_{n+1}), (v_2, v_{n+1}), \ldots, (v_n, v_{n+1})\}$ (with unit weight each, i.e., $w'(e) = 1$ for $e \in E$). Note that the optimal solution is $OPT = 1 + \varepsilon$ associated with the dominating set $\{v_{n+1}\}$. The greedy algorithm, however, generates dominating set $D = \{v_1, \ldots, v_n\}$ with total weight H_n.

4.2 Approximation Algorithm by Randomized Rounding

Observe that the Weighted-Measured Dominating Set problem is equivalent to the following integer program:

Integer Program (IP):

$$\begin{aligned}\min \quad & \sum_{i=1}^{n} w_i x_i \\ \text{s.t.} \quad & \sum_{v_j \in S_i} x_j \geq 1, \ \forall\, 1 \leq i \leq n \\ & x_i \in \{0, 1\}, \ \forall\, 1 \leq i \leq n\end{aligned}$$

Here, x_i represents whether vertex v_i is contained in the dominating set or not. The linear program relaxation is as follows.

Linear Program Relaxation (LPR):

$$\begin{aligned}\min \quad & \sum_{i=1}^{n} w_i x_i \\ \text{s.t.} \quad & \sum_{v_j \in S_i} x_j \geq 1, \ \forall\, 1 \leq i \leq n \\ & x_i \in [0, 1], \ \forall\, 1 \leq i \leq n\end{aligned}$$

Let $(x_1^*, \ldots, x_n^*)$ be the optimal solution of IP, and $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be the optimal solution of LPR. Therefore, we have

$$OPT = \sum_{i=1}^{n} w_i x_i^* \geq \sum_{i=1}^{n} w_i \widetilde{x}_i. \tag{3}$$

Note that LPR is polynomial-time solvable, hence we may compute $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ efficiently, and the "feasible" solution of IP is defined by randomized rounding as follows.

$$x_i = \begin{cases} 1 & \text{with probability } \widetilde{x}_i \\ 0 & \text{otherwise.} \end{cases}$$

Define $D = \{v_i \mid x_i = 1, i = 1, \dots, n\}$. In the following, we analyze the probability that D is a feasible solution of IP. For any vertex v_i, due to the feasibility of $\widetilde{x}_i$'s, the probability that v_i is dominated is

$$1 - \prod_{v_j \in S_i} (1 - \widetilde{x}_j) \geq 1 - (1 - \frac{1}{|S_i|})^{|S_i|} \geq 1 - \frac{1}{e}. \tag{4}$$

That is, the probability is lower bounded by constant $1 - \frac{1}{e}$. To increase this probability, we may use amplification approach to independently execute randomized rounding t times, whose value will be determined later. In this case, for any vertex v_i, the following equality holds:

$$Prob[x_i = 0] \;=\; (1 - \widetilde{x}_i)^t. \tag{5}$$

Let A_i denote the event that v_i is not dominated, due to FKG inequality [1], it is easy to see the following two inequalities hold:

$$Prob[A_i \cap A_j] \;\geq\; Prob[A_i] \cdot Prob[A_j], \tag{6}$$

$$Prob[\overline{A}_i \cap \overline{A}_j] \;\geq\; Prob[\overline{A}_i] \cdot Prob[\overline{A}_j]. \tag{7}$$

Let $B = \bigcap_{i=1}^n \overline{A}_i$ be the event that all vertices are dominated. From (3)-(7), the expected value of the solution produced by randomized rounding, given that event B happens, is

$$\begin{aligned}
E\left[\sum_{i=1}^n w_i x_i \mid B\right] &= \sum_{i=1}^n w_i \cdot Prob\left[x_i = 1 \mid B\right] \\
&= \sum_{i=1}^n w_i \cdot \frac{Prob\left[B \mid x_i = 1\right]}{Prob[B]} \cdot Prob[x_i = 1] \\
&= \sum_{i=1}^n w_i \cdot \frac{Prob\left[\bigcap_{j \notin S_i} \overline{A}_j\right]}{Prob\left[\bigcap_{j=1}^n \overline{A}_j\right]} \cdot Prob[x_i = 1] \\
&\overset{(7)}{\leq} \sum_{i=1}^n w_i \cdot \frac{1}{Prob\left[\bigcap_{j \in S_i} \overline{A}_j\right]} \cdot Prob[x_i = 1] \\
&\overset{(7)}{\leq} \sum_{i=1}^n w_i \cdot \frac{1}{\prod_{j \in S_i} Prob\left[\overline{A}_j\right]} \cdot Prob[x_i = 1] \\
&\overset{(4)}{\leq} \sum_{i=1}^n w_i \cdot \frac{1}{(1 - e^{-t})^{\Delta}} \cdot Prob[x_i = 1]
\end{aligned}$$

$$\stackrel{(5)}{=} \frac{1}{(1-e^{-t})^{\Delta}} \cdot \sum_{i=1}^{n} w_i \cdot (1-(1-\widetilde{x}_i)^t)$$
$$\leq \frac{1}{(1-e^{-t})^{\Delta}} \cdot \sum_{i=1}^{n} w_i \cdot (1-(1-t\widetilde{x}_i))$$
$$\stackrel{(3)}{\leq} \frac{t}{(1-e^{-t})^{\Delta}} \cdot OPT.$$

Defining $t = O(\log \Delta)$, we have

$$E\left[\sum_{i=1}^{n} w_i x_i \mid B\right] \leq O(\log \Delta \cdot OPT).$$

Therefore, we have the following conclusion.

Theorem 4.4. Given any graph G with weighted vertices and edges, with positive probability, the weighted-measured dominating set of G can be computed efficiently within $O(\log \Delta)$ ratio to the optimal solution.

5 CONCLUSIONS

An extension model of Dominating Set problem, Dominating Set with measure functions, is studied in this paper. Specifically, we showed the NP-hardness for the problem with measure function $f(n) = n^{\varepsilon}$, and discussed related approximation algorithms. In addition, the problem we studied in this paper, associated with a number of other extensions of Dominating Set, has many potential applications in wireless Ad Hoc network, in communication network, etc.

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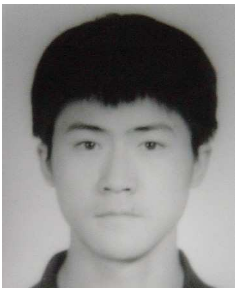

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