Computing and Informatics, Vol. 27, 2008, 297–313

PARTIAL CONVERGENCE AND CONTINUITY OF LATTICE-VALUED POSSIBILISTIC MEASURES

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Manuscript received 19 June 2006; revised 10 November 2006
Communicated by Marián Vajteršic

Abstract. The notion of continuity from above (upper continuity) for lattice-valued possibilistic measures as investigated in [7] has been proved to be a rather strong condition when imposed as demand on such a measure. Hence, our aim will be to introduce some versions of this upper continuity weakened in the sense that the conditions imposed in [7] to the whole definition domain of the possibilistic measure in question will be restricted just to certain subdomains. The resulting notion of partial upper convergence and continuity of lattice-valued possibilistic measures will be analyzed in more detail and some results will be introduced and proved.

Keywords: Partially ordered set, (complete) lattice, set function, lattice-valued possibilistic (possibility) measure, (complete) maxitivity, convergence and continuity from above (upper convergence and continuity), convergence and continuity from below (lower convergence and continuity)

1 INTRODUCTION

The basic idea of real-valued possibilistic measures was conceived by L. A. Zadeh in [9] and possibilistic measures with non-numerical possibility degrees, namely those taking these degrees in partially ordered sets and, more specifically, in complete lattices, were introduced and analyzed in more detail by G. de Cooman in [3]. Since the appearance of these pioneering publications, a lot of work has been done and numerous deep, valuable and interesting results have been achieved in both these fields of possibility (or possibilistic) measures and possibility theory under consideration. It is quite easy to understand that the methods, constructions and results developed, built and achieved when analyzing some formerly investigated set functions like measure theory and probability theory as its particular case as an inspiration and motivation for such investigations often served. Both the properties demonstrating a degree of similarity between probability and possibility measures in certain aspects as well as the properties showing a qualitative difference of both the uncertainty measures under consideration in other aspects are worth being proved and analyzed in more detail.

Continuity from above (upper continuity) and from below (lower continuity) are important properties of set functions which significantly improve the qualities of these functions when applied as mathematical tools for uncertainty quantification and processing. As a matter of fact, the properties of probability and possibility measures related to the notion of continuity are rather different (cf. [7] for some former author's results in this field). A probability measure P defined on a σ-field $\mathcal{A}$ of subsets of a universe Ω of elementary random events is continuous from above, if for each decreasing nested sequence $A_1 \supset A_2 \supset \ldots$ of sets from $\mathcal{A}$ the relation $\lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n)$ holds, and P is continuous from below, if for each increasing nested sequence $A_1 \subset A_2 \subset \ldots$ of sets from $\mathcal{A}$ the relation $\lim_{n\to\infty} P(A_n) = P(\bigcup_{n=1}^{\infty} A_n)$ is valid. As can be easily proved (cf. [6, 8] or another elementary textbook on measure or probability theory), a finitely additive probability measure P on $\mathcal{A}$ is continuous from above if and only if it is continuous from below and this happens if and only if P is a σ-additive (countably additive) probability measure on $\mathcal{A}$.

For possibilistic (we will prefer this adjective in what follows) measures, real-valued as well as the lattice-valued ones, continuity from above and from below are not, in general, equivalent properties due to the asymmetric role of supremum and infimum operations when defining possibilistic measures, and it is the continuity from above which can be seen as a qualitatively stronger property significantly reducing the variety of possibilistic measures obeying this continuity. E.g., also the very simple possibilistic measure Π ascribing the value 0 to the empty subset of an infinite space Ω and the value 1 to each other (i.e., nonempty) subset of Ω is not continuous from above on $\mathcal{P}(\Omega)$, as it is not continuous from above in the empty set (take $\Omega_0 = \{\omega_1, \omega_2, \ldots\} \subset \Omega$ and take $A_n = \{\omega_n, \omega_{n+1}, \ldots\}$, so that $\Pi(A_n) = 1$ for any $n = 1, 2, \ldots$, but $\Pi(\bigcap_{n=1}^{\infty} A_n) = \Pi(\emptyset) = 0$. It is why we intend to investigate, in this paper, also partially continuous possibilistic measures (mostly partially continuous from above) in the sense that the relation $\Pi(\bigcap_{n=1}^{\infty} A_n) = \inf\{\Pi(A_n) : n = 1, 2, \ldots\}$ is valid only for *some* sets $A \subset \Omega$ and *some* sequences $A_1, A_2, \ldots$ such that $\bigcap_{n=1}^{\infty} A_n = A$. On the other side, however, we will extend the notion of convergence also to not necessarily nested and not necessarily countable (i.e., definable as a sequence) systems of subsets of Ω so that we will investigate, in general, the case when the relation $\Pi(\bigcap \mathcal{R}) = \inf\{\Pi(A) : A \in \mathcal{R}\}$ holds for some systems $\mathcal{R}$ of subsets of Ω, here $\bigcap \mathcal{R} = \bigcap_{A\in\mathcal{R}} A$. The dual case when

$\Pi(\bigcup \mathcal{R}) = \sup\{\Pi(A) : A \in \mathcal{R}\}$, $\bigcup \mathcal{R} = \bigcup_{A \in \mathcal{R}} A$, will also be touched, even if the problems concerning the convergence from above will be preferably focused.

2 BASIC NOTIONS, DEFINITIONS AND PRELIMINARIES

The reader is supposed to be familiar with the notion of *partially ordered set (p.o.set)* defined by a pair $\mathcal{T} = \langle T, \leq \rangle$, where T is a nonempty set and $\leq$ is a reflexive, antisymmetric and transitive binary relation on T (subset of the Cartesian product $T \times T$, in the set-theoretic notation). The operations of *supremum* (denoted by $\vee$ or $\bigvee$) and *infimum* (denoted by $\wedge$ or $\bigwedge$) in p.o. set $\mathcal{T}$ are defined in the standard way and it is a well-known fact that, given a subset $A \subset T$, neither the supremum $\bigvee A = \bigvee_{t \in A} t$, nor the infimum $\bigwedge A = \bigwedge_{t \in A} t$ need be defined in general. In order to simplify our reasoning and the resulting mathematical formalization we will suppose, throughout this work, that p.o.set $\mathcal{T} = \langle T, \leq \rangle$ satisfies the condition that $\bigvee A$ and $\bigwedge A$ are defined (hence, are elements of T) for each $A \subset T$. In this case, p.o.set $\mathcal{T} = \langle T, \leq \rangle$ is called *complete lattice*. Consequently, also the elements $\bigwedge T$, denoted by $\oslash_{\mathcal{T}}$ and called the *zero (element)* of $\mathcal{T}$, and $\bigvee T$, denoted by $\mathbf{1}_{\mathcal{T}}$ and called the *unit (element)* of $\mathcal{T}$ are defined and the conventions $\bigwedge \emptyset = \mathbf{1}_{\mathcal{T}}$ and $\bigvee \emptyset = \oslash_{\mathcal{T}}$ for empty subset of T are applied. Because of our intention to apply complete lattices as structures in which uncertainty degrees take their values, it is perhaps worth recalling explicitly that both the p.o.sets $\langle [0,1], \leq \rangle$, i.e., the unit interval of real numbers equipped by their standard linear ordering, and $\langle \mathcal{P}(X), \subset \rangle$, i.e., the power-set of all subsets of a nonempty set X partially ordered by the relation of set-theoretic inclusion (complete Boolean algebra, as a matter of fact) are complete lattices. Moreover, complete lattice seems to be the most specific structure still common for both these most often used structures for uncertainty quantification and processing.

The reader is recommended to consult [2, 4], or some more recent textbook or monograph when seeking for more details concerning p.o.sets and lattices.

The idea to consider complete lattices as structures in which uncertainty degrees, in particular membership functions of fuzzy sets, take their values, originates from J. A. Goguen [5] and was applied to possibilistic measures and investigated in detail by G. de Cooman in [3]. In these references, as well as in a number of other ones, also the philosophical and methodological issues involved by lattice-valued uncertainty degrees are analyzed and discussed, so that we purposedly omit these aspects here and begin our considerations with a very general, but fitted for our further purposes, definition of lattice-valued possibilistic measures.

In what follows, given a nonempty set Ω, we denote by $\|A\|$ ($\|\mathcal{S}\|$, $\|\tilde{\mathcal{S}}\|$, ...) the cardinality (cardinal number) of a subset A of Ω (of a system $\mathcal{S}$ of subsets of Ω, i.e., of $\mathcal{S} \subset \mathcal{P}(\Omega)$, of a system $\tilde{\mathcal{S}}$ of systems of subsets of Ω, i.e., of $\tilde{\mathcal{S}} \subset \mathcal{P}\,(\mathcal{P}(\Omega))$, etc.).

Definition 1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\mathcal{A}$ be a system of subsets of Ω containing the empty set $\emptyset$ and the space Ω

($\{\emptyset, \Omega\} \subset \mathcal{A} \subset \mathcal{P}(\Omega)$, in symbols), let α be a cardinal number over the universe Ω. The mapping $\Pi : \mathcal{A} \to T$ is called *α-maxitive $\mathcal{T}$-(valued) possibilistic measure on $\mathcal{A}$,* if $\Pi(\emptyset) = \oslash_{\mathcal{T}}$, $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$, and

$$\Pi\left(\bigcup \mathcal{R}\right) = \bigvee\{\Pi(A) : A \in \mathcal{R}\} \tag{1}$$

holds for each $\mathcal{R} \subset \mathcal{A}$ such that $\|\mathcal{R}\| \leq \alpha$ and $\bigcup \mathcal{R} = \bigcup_{A \in \mathcal{R}} A \in \mathcal{A}$ holds.

If $\mathcal{A} = \mathcal{P}(\Omega)$ and $\alpha = 2$, then this definition reduces to the standard definition of $\mathcal{T}$-valued possibilistic measure on $\mathcal{P}(\Omega)$, if $\mathcal{A} = \mathcal{P}(\Omega)$ and $\alpha = 2^{\|\Omega\|}$, then we arrive at the definition of complete $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Some more or less trivial consequences of this definition will be presented and discussed below, but it is perhaps worth being noted explicitly that if $\mathcal{A} \neq \mathcal{P}(\Omega)$, i.e., if Π is a *partial* mapping on $\mathcal{P}(\Omega)$, then α-maxitivity of Π for some $\alpha \in \mathcal{N}^+ = \{1, 2, \ldots\}$ does not imply, in general, β-maxitivity for $\beta > \alpha, \beta \in \mathcal{N}^+$.

Indeed, let $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ be a disjoint covering of Ω by nonempty sets Ω_i, $i = 1, \ldots, 4$, let $\mathcal{A} = \{\emptyset, \Omega_1, \Omega_2, \Omega_3, (\Omega_1 \cup \Omega_2 \cup \Omega_3), \Omega\}$, let $\Pi(\emptyset) = \oslash_{\mathcal{T}} < \Pi(\Omega_1)$, $\Pi(\Omega_2)$, $\Pi(\Omega_3) \leq t < \Pi(\Omega_1 \cup \Omega_2 \cup \Omega_3) \leq \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$. Then $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$ holds only for the systems $\{\emptyset, B\}, B \in \mathcal{A}, \{\Omega, B\}, B \in \mathcal{A}$, and $\{\Omega_i, (\Omega_1 \cup \Omega_2 \cup \Omega_3)\}$, $i = 1, 2, 3$. In all these cases the relation $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ holds, as may be easily checked, so that Π defines a 2-maxitive $\mathcal{T}$-possibilistic measure on $\mathcal{A}$. However, the relation $\Pi(\Omega_1 \cup \Omega_2 \cup \Omega_3) = \Pi(\Omega_1) \vee \Pi(\Omega_2) \vee \Pi(\Omega_3)$ does not hold and need not hold, as neither $\Pi(\Omega_1 \cup \Omega_2)$, nor $\Pi(\Omega_2 \cup \Omega_3)$, nor $\Pi(\Omega_1 \cup \Omega_3)$ are defined, so that $\Omega_1 \cup \Omega_2 \cup \Omega_3$ cannot be expressed as the union of *two* sets from the definition domain $\mathcal{A}$ of Π.

According to Definition 1, when considering a $\mathcal{T}$-possibilistic measure defined on a system $\mathcal{A}$ of subsets of Ω, we always tacitly assume that $\mathcal{A}$ contains $\emptyset$ and Ω.

Definition and processing of a real-valued possibilistic measure Π becomes significantly simplified, if this possibilistic measure possesses the (possibilistic) distribution. This is the case when all singletons $\{\omega\}$, $\Omega \in \Omega$, are in the definition domain $\mathcal{A} \subset \mathcal{P}(\Omega)$ of Π and the identity $\Pi(A) = \bigvee\{\Pi(\{\omega\}) : \omega \in A\}$ holds for each $A \in \mathcal{A}$. As Ω is supposed to be in $\mathcal{A}$, the relation $\bigvee\{\Pi(\{\omega\}) : \omega \in \Omega\} = 1$ follows and Π can be easily and uniquely extended from $\mathcal{A}$ to $\mathcal{P}(\Omega)$, simply taking the identity for $\Pi(A)$ from above as the definition of $\Pi(A)$ for the subsets of Ω outside of $\mathcal{A}$. Consequently, each complete real-valued possibilistic measure on $\mathcal{P}(\Omega)$ possesses the possibilistic distribution. Sometimes real-valued possibilistic distributions on Ω, i.e., mappings $\pi : \Omega \to [0, 1]$ such that $\bigvee\{\pi(\omega) : \omega \in [0, 1]\} = 1$, are taken as the basic stones when introducing possibilistic measures on $\mathcal{P}(\Omega)$, taking the relation $\Pi(A) = \bigvee\{\pi(\omega) : \omega \in A\}$ as definition. On the other side, when introducing real-valued possibilistic measures on $\mathcal{A} \subset \mathcal{P}(\Omega)$ axiomatically, i.e., as normalized real-valued 2-maxitive set functions, there exist incomplete possibilistic measures on the power-set $\mathcal{P}(\Omega)$ of an infinite space Ω, not possessing possibilistic distribution. Indeed, let us recall the well-known example when $\Pi(A) = 0$, if A is empty or finite, and $\Pi(A) = 1$ for infinite subsets of an infinite space Ω.

Also in the case of lattice-valued possibilistic measures to which our attention in this paper is focused, lattice-valued possibilistic distributions will play an important role. Our definition will be introduced at rather abstract and general level, relativizing or better graduating the notion of possibilistic distribution in a way similar to that in which we graduated the notion of maxitivity in Definition 1.

Definition 2. Let $\mathcal{T}$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure defined on $\mathcal{A} \subset \mathcal{P}(\Omega)$, let $\mathcal{R} \subset \mathcal{A}$ be a nonempty subsystem of $\mathcal{A}$. The possibilistic measure Π *possesses $\mathcal{R}$-(possibilistic) distribution,* if for each $A \in \mathcal{A}$ the relation

$$\Pi(A) = \bigvee \{\Pi(B) : B \in \mathcal{R}, B \subset A\} \tag{2}$$

holds.

As Π is a monotone measure on $\mathcal{A}$, $\Pi(B) \leq \Pi(A)$ holds for each $B \subset A$, $A, B \in \mathcal{A}$, so that the inequality $\Pi(A) \geq \bigvee\{\Pi(B) : B \in \mathcal{R}, B \subset A\}$ is obviously valid for each $A \in \mathcal{A}$. Hence, each $\mathcal{T}$-possibilistic measure Π on $\mathcal{A} \subset \mathcal{P}(\Omega)$ evidently possesses $\mathcal{A}$-distribution, as in this case for each $A \in \mathcal{A}$ A is among the subsets for which the supremum value is taken, so that equality (2.2) holds. If $\mathcal{R}$ is the system of all singletons of $\mathcal{P}(\Omega)$, then $\mathcal{R}$-distribution is nothing else than the standard $\mathcal{T}$-possibilistic distribution. In what follows, our idea will be to use, in the role of $\mathcal{R}$, the systems of sets containing just the sets of cardinality not exceeding a given threshold value, so keeping in mind that distribution should enable to reduce the definition and calculation of possibility degrees ascribed to "large" sets to the application of supremum operation to the values ascribed to "small" sets.

Let us reconsider the simple example of possibilistic measure on $\mathcal{P}(\Omega)$ not possessing the distribution, now in the lattice-valued setting. So, let Ω be an infinite space, let Π be the possibilistic measure defined on $\mathcal{P}(\Omega)$ when setting $\Pi(A) = \oslash_{\mathcal{T}}$, if A is empty or finite, $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ otherwise. As can be easily checked, Π is an α-maxitive $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$ for each $\alpha = 1, 2, \ldots$ Let $\mathcal{R}_\alpha = \{A \subset \Omega : \|A\| \leq \alpha\}$, $\alpha = 1, 2, \ldots$ let $\mathcal{R}_\infty = \{A \subset \Omega : \|A\| = \infty\}$. Then Π possesses $\mathcal{R}_\alpha$-distribution for *no* $\alpha = 1, 2, \ldots$ Indeed, for each infinite $A \subset \Omega$, $\Pi(A) = \mathbf{1}_{\mathcal{T}} \neq \bigvee\{\Pi(B) : B \in \mathcal{R}_\alpha, A \subset A\} = \oslash_{\mathcal{T}}$, as $\Pi(B) = \oslash_{\mathcal{T}}$ for each $B \in \mathcal{R}_\alpha$. However, Π possesses $\mathcal{R}_\infty$-distribution. If A is finite, then

$$\Pi(A) = \oslash_{\mathcal{T}} = \bigvee\{\Pi(B) : B \in \mathcal{R}_\infty, B \subset A\} = \bigvee \emptyset, \tag{3}$$

as there is no infinite subset of A so that the convention for $\bigvee \emptyset$ applies. If A is infinite, an infinite subset of A exists, at least A itself, so that $\bigvee\{\Pi(B) : B \in \mathcal{R}_\infty, B \subset A\} = \Pi(A)$; hence, Π possesses $\mathcal{R}_\infty$-distribution.

3 SOME SIMPLE RESULTS ON α-MAXITIVITY AND $\mathcal{R}$-DISTRIBUTIONS

The following assertions are almost self-evident, but they are perhaps worth being stated explicitly.

Lemma 1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty space.

(i) Let Π be an α-maxitive $\mathcal{T}$-possibilistic measure on $\mathcal{A} \subset \mathcal{P}(\Omega)$. Then Π is β-maxitive on $\mathcal{A}$ for every $\beta \leq \alpha$.

(ii) Let $\mathcal{A} = \mathcal{P}(\Omega)$, let Π be 2-maxitive on $\mathcal{P}(\Omega)$. Then Π is n-maxitive on $\mathcal{P}(\Omega)$ for every $n = 1, 2, \ldots$

(iii) If Π on $\mathcal{A}$ possesses $\mathcal{R}$-distribution, then Π possesses $\mathcal{S}$-distribution for every $\mathcal{R} \subset \mathcal{S} \subset \mathcal{P}(\Omega)$. In particular,

(iv) if Π possesses $\mathcal{R}_\alpha(\Omega)$-distribution, where $\mathcal{R}_\alpha(\Omega) = \{A \subset \Omega, \|A\| \leq \alpha\}$, then Π possesses $\mathcal{R}_\beta(\Omega)$-distribution for every $\beta \geq \alpha$.

Proof. (i) is obvious, (ii) can be easily proved by induction. Indeed, for $n = 1$ and 2 the assertion is trivial; suppose that it holds for some $n \geq 2$, i.e., let Π be n-maxitive. For every $(n+1)$-tuple $A_1, A_2, \ldots, A_n, A_{n+1}$ of subsets of Ω we obtain

$$\begin{aligned} \Pi\left(\bigcup_{i=1}^{n+1} A_i\right) &= \Pi\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right) = \Pi\left(\bigcup_{i=1}^{n} A_i\right) \vee \Pi(A_{n+1}) = \\ &= \left(\bigvee_{i=1}^{n} \Pi(A_i)\right) \vee \Pi(A_{n+1}) = \bigvee_{i=1}^{n+1} \Pi(A_i), \end{aligned} \tag{4}$$

so that Π is $(n+1)$-maxitive and (ii) is proved. For no matter which system $\mathcal{S}$ of subsets of a subset A of Ω we obtain that $\Pi(B) \leq \Pi(A)$ holds for each $B \in \mathcal{S}$; hence, $\bigvee\{\Pi(B) : B \in \mathcal{S}\} \leq \Pi(A)$ follows. Consequently, as Π possesses $\mathcal{R}$-distribution and $\mathcal{R} \subset \mathcal{S}$ is supposed to hold, we obtain that

$$\Pi(A) = \bigvee\{\Pi(B) : B \subset A, B \in \mathcal{R}\} \leq \bigvee\{\Pi(B) : B \subset A, B \in \mathcal{S}\} \leq \Pi(A) \tag{5}$$

holds, so that Π possesses $\mathcal{S}$-distribution and (iii) is proved. (iv) is just a particular case of (iii), as $\mathcal{R}_\alpha(\Omega) \subset \mathcal{R}_\beta(\Omega)$ obviously holds for each $\alpha \leq \beta$. □

Lemma 2. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice; let Π be $\|\Omega\|$-maxitive $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Then Π possesses $\mathcal{R}_1(\Omega)$-distribution and it is $2^{\|\Omega\|}$-maxitive; hence, it is complete in the standard sense.

Proof. For each $A \subset \Omega, A = \bigcup_{\omega \in A}\{\omega\}$ so that, due to the $\|\Omega\|$-maxitivity of Π, $\Pi(A) = \bigvee_{\omega \in A} \Pi(\{\omega\})$; hence, Π possesses $\mathcal{R}_1(\Omega)$-distribution. For any $\mathcal{R} \subset \mathcal{P}(\Omega)$,

$$\begin{aligned} \Pi\left(\bigcup \mathcal{R}\right) &= \bigvee\left\{\Pi(\{\omega\}) : \omega \in \bigcup \mathcal{R}\right\} = \bigvee_{A \in \mathcal{R}} \left(\bigvee_{\omega \in A} \Pi(\{\omega\})\right) = \\ &= \bigvee\{\Pi(A) : A \in \mathcal{R}\}, \end{aligned} \tag{6}$$

so that Π is $2^{\|\Omega\|}$-maxitive. Finally, Π is α-maxitive for each $\alpha > 2^{\|\Omega\|}$ as there are no subsystems of $\mathcal{P}(\Omega)$ of cardinality greater than $2^{\|\Omega\|}$. Hence, Π is α-maxitive for every cardinal number α (maxitivity for $\beta \leq \alpha$ follows from Lemma 1(i)). □

Theorem 1. Let $\mathcal{T}$ be a complete lattice, let Π be an α-maxitive $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$ possessing $\mathcal{R}_\alpha(\Omega)$-distribution. Then Π is β-maxitive for every cardinal number β and possesses $\mathcal{R}_1(\Omega)$-distribution.

Proof. As Π is α-maxitive, for every $A \subset \Omega$ such that $\|A\| \leq \alpha$ holds; consequently, for every $A \in \mathcal{R}_\alpha(\Omega)$, the relation $\Pi(A) = \bigvee\{\Pi(\{\omega\}) : \omega \in A\}$ holds. If $A \subset \Omega$, $\|A\| > \alpha$ is the case, then we apply the $\mathcal{R}_\alpha(\Omega)$-distribution of Π and we obtain that

$$\begin{aligned} \Pi(A) &= \bigvee\{\Pi(B) : B \subset A, B \in \mathcal{R}_\alpha(\Omega)\} = \\ &= \bigvee\left\{\left(\bigvee\{\Pi(\{\omega\}) : \omega \in B\}\right) : B \subset A, B \in \mathcal{R}_\alpha(\Omega)\right\} = \\ &= \bigvee\left\{\Pi(\{\omega\}) : \omega \in \bigcup\{B : B \subset A, B \in \mathcal{R}_\alpha(\Omega)\}\right\}. \end{aligned} \tag{7}$$

As $\{\omega\} \in \mathcal{R}_\alpha(\Omega)$ and $\{\omega\} \subset A$ holds for each $\omega \in A$, the relation

$$\bigcup\{B : B \subset A, B \in \mathcal{R}_\alpha(\Omega)\} = A$$

follows, so that $\Pi(A) = \bigvee\{\Pi(\{\omega\}) : \omega \in A\}$ holds and Π is $\|\Omega\|$-maxitive. Due to Lemma 2, Π is maxitive for every cardinal number β. □

It is perhaps worth being noted explicitly that for a $\mathcal{T}$-possibilistic measure Π possessing $\mathcal{R}_\alpha(\Omega)$-distribution the maxitivity of Π for any $\beta < \alpha$ is not sufficient in order to prove that Π possesses $\mathcal{R}_1(\Omega)$-distribution and is β-maxitive for every cardinal number β (as Theorem 1 claims). Indeed, consider the example from above with an infinite countable set Ω and with Π ascribing the value $\emptyset_\mathcal{T}$ to the empty set and to finite subsets of Ω, and ascribing $\mathbf{1}_\mathcal{T}$ to infinite subsets. This $\mathcal{T}$-possibilistic measure is defined on $\mathcal{P}(\Omega)$, it is γ-maxitive for every cardinal number smaller than $\aleph_0$, i.e., for every $\gamma = 1, 2, \ldots, \Pi$ possesses the $\mathcal{R}_\alpha(\Omega)$-distribution for $\alpha = \aleph_0$, but Π does not possess $\mathcal{R}_1(\Omega)$-distribution, i.e., the possibilistic distribution in the standard sense.

4 WEAKENING THE NOTION OF CONTINUITY OF LATTICE-VALUED POSSIBILISTIC MEASURES

As analyzed in more detail for real-valued possibilistic measures in [1] and for lattice-valued ones in [7], the question whether a possibilistic measure can be defined by its possibilistic distribution is closely related to the property of continuity from above and from below of the possibilistic measure in question. Like as in the case of maxitivity and possibilistic distributions considered above, our aim will be to weaken the notion of continuity of possibilistic measures by reducing this property just to *some* of the nested sequences of sets to which the standard general definition applies. Let us recall this definition.

Definition 3. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure on a nonempty σ-field $\mathcal{A}$ of subsets of Ω.

(i) Π is *continuous from above on* $\mathcal{A}$, if for each nested sequence $A_1 \supset A_2 \supset \ldots$ of sets from $\mathcal{A}$ the relation

$$\Pi\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigwedge_{i=1}^{\infty} \Pi(A_i) \tag{8}$$

holds.

(ii) Π is *continuous from below on* $\mathcal{A}$, if for each nested sequence $A_1 \subset A_2 \subset \ldots$ of sets from $\mathcal{A}$ the relation

$$\Pi\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} \Pi(A_i) \tag{9}$$

holds (being a σ-field, $\mathcal{A}$ contains $\bigcap_{i=1}^{\infty} A_i$ as well as $\bigcup_{i=1}^{\infty} A_i$).

If Π is defined on a σ-field $\mathcal{A} \subset \mathcal{P}(\Omega)$ and if Π is continuous from below on $\mathcal{A}$ in the sense of (4.2), then Π is $\aleph_0$-maxitive on $\mathcal{A}$. Indeed, let $\mathcal{R} \subset \mathcal{A}$ be such that $\|\mathcal{R}\| = \aleph_0$. Then there exists a one-to-one enumeration of sets in $\mathcal{R}$ by positive integers, i.e., $\mathcal{R} = \{A_1, A_2, \ldots\}$. Setting $B_n = \bigcup_{i=1}^{n} A_i$ for each $n = 1, 2, \ldots$, we obtain that $B_n \subset B_{n+1}$ holds so that, as Π is continuous from below, we obtain

$$\begin{aligned}\Pi\left(\bigcup_{i=1}^{\infty} A_i\right) &= \Pi\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigvee_{i=1}^{\infty} \Pi(B_i) = \bigvee_{i=1}^{\infty} \Pi\left(\bigcup_{j=1}^{i} A_j\right) = \\ &= \bigvee_{i=1}^{\infty}\left(\bigvee_{j=1}^{i} \Pi(A_j)\right) = \bigvee_{j=1}^{\infty} \Pi(A_j),\end{aligned} \tag{10}$$

so that Π is $\aleph_0$-maxitive. A dual assertion for Π continuous from above on $\mathcal{A}$ does not hold in general. Indeed, let $\mathcal{R} = \{A_1, A_2, \ldots\} \subset \mathcal{A}$ be as above, set $B_n = \bigcap_{i=1}^{n} A_i$ for each $n = 1, 2, \ldots$ As Π is continuous from above on $\mathcal{A}$, we obtain

$$\Pi\left(\bigcap_{i=1}^{\infty} A_i\right) = \Pi\left(\bigcap_{i=1}^{\infty} B_i\right) = \bigwedge_{i=1}^{\infty} \Pi(B_i) = \bigwedge_{i=1}^{\infty} \Pi\left(\bigcap_{j=1}^{i} A_j\right). \tag{11}$$

However, as $\bigcap_{j=1}^{i} A_j = A_i$ need not hold for a non-nested sequence $A_1, A_2, \ldots$ of subsets of Ω and also the identity $\Pi(\bigcap_{j=1}^{i} A_j) = \bigwedge_{j=1}^{i} \Pi(A_j)$ need not be the case in general, the only consequence which can be deduced from 11 is the obvious inequality $\Pi(\bigcap_{i=1}^{\infty} A_i) \leq \bigwedge_{i=1}^{\infty} \Pi(A_i)$. Hence, in order to obtain the equality in this case, we have to strengthen the definition of continuity from above, simply imposing the equality $\Pi(\bigcap_{i=1}^{\infty} A_i) = \bigwedge_{i=1}^{\infty} \Pi(A_i)$ as axiomatic demand to be satisfied for each (not necessarily nested) sequence $A_1, A_2, \ldots$ of sets from $\mathcal{A}$. Another advantage of this stronger definition consists in the fact that it can be applied also to non-countable systems $\mathcal{R} \subset \mathcal{A}$, as no enumeration of sets from $\mathcal{R}$ is necessary. So, we arrive at this very general definition.

Definition 4. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure defined on a nonempty σ-field of subsets of a nonempty space Ω, let α, β be cardinal numbers over Ω.

(i) Π is called $\langle \alpha, \beta \rangle$- *continuous from above on* $\mathcal{A}$, if for each $\mathcal{R} \subset \mathcal{A}$ such that $\|\mathcal{R}\| \leq \alpha$ and $\|\bigcap \mathcal{R}\| \geq \beta$ holds, the equality

$$\Pi\left(\bigcap \mathcal{R}\right) = \bigwedge \{\Pi(A) : A \in \mathcal{R}\} \tag{12}$$

is valid, here $\bigcap \mathcal{R} = \bigcap_{A \in \mathcal{R}} A$.

(ii) Π is called $\langle \alpha, \beta \rangle$- *continuous from below* on $\mathcal{A}$, if for each $\mathcal{R} \subset \mathcal{A}$ such that $\|\mathcal{R}\| \leq \alpha$ and $\|A\| \geq \beta$ for each $A \in \mathcal{R}$ holds, the equality

$$\Pi\left(\bigcup \mathcal{R}\right) = \bigvee \{\Pi(A) : A \in \mathcal{R}\} \tag{13}$$

is valid, here $\bigcup \mathcal{R} = \bigcup_{A \in \mathcal{R}} A$.

(iii) Π is called β-*continuous from above (or from below) on* $\mathcal{A}$, if it is $\langle \|\mathcal{A}\|, \beta \rangle$-continuous from above (or from below) on $\mathcal{A}$, i.e., if (4.5) or (4.6) holds for every $\mathcal{R} \subset \mathcal{A}$.

The following consequence of this definition is almost self-evident but perhaps worth being introduced explicitly.

Lemma 3. Let $\mathcal{T}, \Pi, \alpha, \beta$ be as in Definition 4.2. If Π is $\langle \alpha, \beta \rangle$-continuous from above (or from below) on $\mathcal{A}$, it is also $\langle \alpha_1, \beta_1 \rangle$-continuous from above (or from below) on $\mathcal{A}$ for every $\alpha_1 \leq \alpha$ and $\beta_1 \geq \beta$.

Proof. Set

$$\rho(\alpha, \beta) = \left\{\mathcal{R} \subset \mathcal{P}(\Omega) : \|\mathcal{R}\| \leq \alpha, \|\bigcap \mathcal{R}\| \geq \beta\right\}, \tag{14}$$

so that Definition (4.2) (i) reads that for each $\mathcal{R} \in \rho(\alpha, \beta)$ (4.5) holds. If $\alpha_1 \leq \alpha$ and $\beta_1 \geq \beta$ is the case, then $\rho(\alpha_1, \beta_1) \subset \rho(\alpha, \beta)$ holds, so that for $\mathcal{R} \in \rho(\alpha_1, \beta_1)$ (4.5) trivially follows. For (ii) and (4.6) the situation is analogous, just with

$$\rho(\alpha, \beta) = \{\mathcal{R} \subset \mathcal{P}(\Omega) : \|\mathcal{R}\| \leq \alpha, \|A\| \geq \beta \text{ for every } A \in \mathcal{R}\}. \tag{15}$$

Again, for $\alpha_1 \leq \alpha$ and $\beta_1 \geq \beta$ the inclusion $\rho(\alpha_1, \beta) \subset \rho(\alpha, \beta\rangle$ is valid, so that for each $\mathcal{R} \in \rho(\alpha_1, \beta_1)$ (4.6) follows. □

The notion of $\langle \alpha, \beta \rangle$-continuity of $\mathcal{T}$-possibilistic measures offers a rather flexible tool when classifying the properties of such measures. E.g., for each $n = 0, 1, \dots$ there exists a finite or infinite space Ω and a $\mathcal{T}$-possibilistic measure Π on $\mathcal{P}(\Omega)$ which is $(n+1)$-continuous, but not n-continuous, from above on $\mathcal{P}(\Omega)$. The reasoning is as follows: take Ω such that $\|\Omega\| > n + 1$ holds and take a proper subset $\Omega_0 \subset \Omega$ such that $\|\Omega_0\| = n$. Consider the mapping $\pi : \Omega \to T$ such that $\pi(\omega) = \oslash_{\mathcal{T}}$, if $\omega \in \Omega_0, \pi(\omega) = \mathbf{1}_{\mathcal{T}}$ otherwise. Obviously, as $\Omega - \Omega_0 \neq \emptyset, \bigvee_{\omega \in \Omega} \pi(\omega) = \mathbf{1}_{\mathcal{T}}$, so that π defines a $\mathcal{T}$-possibilistic distribution on Ω. Setting $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$ for every $A \subset \Omega$, we obtain a complete $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Indeed, as can be easily seen, $\Pi(A) = \mathbf{1}_{\mathcal{T}}$, if $A \cap (\Omega - \Omega_0) \neq \emptyset, \Pi(A) = \oslash_{\mathcal{T}}$, if $A \subset \Omega_0$. For each $\mathcal{R} \subset \mathcal{P}(\Omega)$, $\Pi(\bigcap \mathcal{R}) = \mathbf{1}_{\mathcal{T}}$ iff $\bigcup \mathcal{R} \cap (\Omega - \Omega_0) \neq \emptyset$, but this is the case just when there exists $A \in \mathcal{R}$ such that $A \cap (\Omega - \Omega_0) \neq \emptyset$ and, consequently, $\Pi(A) = \bigvee\{\Pi(B) : B \in \mathcal{R}\} = \mathbf{1}_{\mathcal{T}}$ holds. On the other side, $\Pi(\bigcup \mathcal{R}) = \oslash_{\mathcal{T}}$ iff $\bigcup \mathcal{A} \subset \Omega_0$ and, consequently, $A \subset \Omega_0$ for each $A \in \mathcal{R}$ holds, but in this case $\Pi(\bigcup \mathcal{R}) = \oslash_{\mathcal{T}} = \bigvee\{\Pi(A) : A \in \mathcal{R}\}$ holds again. Consequently, Π is a complete ($2^{\|\Omega\|}$-maxitive, in our terms) $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$.

Moreover, Π is $(n+1)$-continuous from above (i.e., $\langle \alpha, n+1 \rangle$-continuous from above for every $\alpha \leq \|\mathcal{P}(\Omega]\| = 2^{\|\Omega]}$), as can be easily proved. Let $\mathcal{R} \subset \mathcal{P}(\Omega)$ be such that $\|\bigcap \mathcal{R}\| \geq n+1$ holds, then $\|A\| \geq n+1$ for each $A \in \mathcal{R}$ follows. Consequently, each $A \in \mathcal{R}$ contains an element from $\Omega - \Omega_0$, so that $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ and the equality $\bigwedge\{\Pi(A) : A \in \mathcal{R}\} = \Pi(\bigcap \mathcal{R}) = \mathbf{1}_{\mathcal{T}}$ is valid, hence, Π is $(n+1)$-continuous from above. However, as $\|\Omega\| > n+1$ holds, there exist at least two different elements ω_1, ω_2 in $\Omega - \Omega_0$. Take $A_1 = \Omega_0 \cup \{\omega_1\}$, $A_2 = \Omega_0 \cup \{\omega_2\}$, $\mathcal{R} = \{A_1, A_2\}$, then $\Pi(\bigcap \mathcal{R}) = \Pi(A_1 \cap A_2) = \Pi(\Omega_0) = \oslash_{\mathcal{T}}$, but $\Pi(A_1) = \Pi(A_2) = \mathbf{1}_{\mathcal{T}}$, so that $\Pi(\bigcap \mathcal{R}) \neq \Pi(A_1) \wedge \Pi(A_2)$ and Π is not n-continuous from above on $\mathcal{P}(\Omega)$.

The already mentioned possibilistic measure ascribing the value $\oslash_{\mathcal{T}}$ to the empty set and to finite subsets of an infinite space Ω, and the value $\mathbf{1}_{\mathcal{T}}$ to infinite subsets, is obviously $\aleph_0$-continuous from above on $\mathcal{P}(\Omega)$ but not n-continuous from above on $\mathcal{P}(\Omega)$, no matter which the $n = 1, 2, \ldots$ may be. If $\mathcal{R} \subset \mathcal{P}(\Omega)$ is such that $\|\bigcap \mathcal{R}\| \geq \aleph_0$ holds, then $\|A\| \geq \aleph_0$ holds for every $A \in \mathcal{R}$, so that the relation $\Pi(\bigcap \mathcal{R}) = \mathbf{1}_{\mathcal{T}} = \bigwedge\{\Pi(A) : A \in \mathcal{R}\}$ follows. On the other hand, taking a countable subset $\Omega_0 = \{\omega_1, \omega_2, \ldots\} \subset \Omega$ and, given n finite, setting $A_{n,k} = \{\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+k+1}, \omega_{n+k+z}, \ldots\}$, we obtain that $\bigcap_{k=1}^{\infty} A_{n,k} = \{\omega_1, \ldots, \omega_n\}$. Hence, $\|\{\omega_1, \ldots, \omega_n\}\| = \|\bigcap_{k=1}^{\infty} A_{n,k}\| \geq n$ is the case, $\Pi(\bigcap_{k=1}^{\infty} A_{n,k}) = \oslash_{\mathcal{T}}$, but $\Pi(A_{n,k}) = \mathbf{1}_{\mathcal{T}}$ for each k, so that Π is not n-continuous from above on $\mathcal{P}(\Omega)$.

A simple generalization of this construction may read as follows. Let Ω be an infinite space, let $\Pi : \mathcal{P}(\Omega) \to T$ be such that $\Pi(A) = \mathbf{1}_{\mathcal{T}}$, if $\|A\| = \|\Omega\|$, $\Pi(A) = \oslash_{\mathcal{T}}$ otherwise, i.e., if $\|A\| < \|\Omega\|$ is the case. As can be easily verified, Π is a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Indeed, for each $A, B \subset \Omega$, $\Pi(A \cup B) = \oslash_{\mathcal{T}}$ iff $\|A \cup B\| < \|\Omega\|$ holds, but in this case $\|A\| < \|\Omega\|$ and $\|B\| < \|\Omega\|$ follows, so that $\Pi(A) = \Pi(B) = \oslash_{\mathcal{T}}$ follows as well. If $\Pi(A \cup B) = \mathbf{1}_{\mathcal{T}}$ is the case, then $\|A \cup B\| = \|\Omega\|$ holds, but in this case either $\|A\| = \|\Omega\|$ or $\|B\| = \|\Omega\|$ must be valid, so that the relation $\Pi(A \cup B) = \mathbf{1}_{\mathcal{T}} = \Pi(A) \vee \Pi(B)$ holds again. Moreover, Π is $\|\Omega\|$-continuous from above on $\mathcal{P}(\Omega)$. Take $\mathcal{R} \subset \mathcal{P}(\Omega)$ such that $\|\bigcap \mathcal{R}\| = \|\Omega\|$ holds, then $\|A\| = \|\Omega\|$ and, consequently, $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ for every $A \in \mathcal{R}$ follows, so that $\Pi(\bigcap \mathcal{R}) = \bigwedge\{\Pi(A) : A \in \mathcal{R}\}$ is the case.

However, take $\alpha < \|\Omega\|$, take $\Omega_0 \subset \Omega$ such that $\|\Omega_0\| = \alpha$, so that $\|\Omega - \Omega_0\| = \|\Omega\|$ holds. For $\mathcal{R} = \{A_\omega : A_\omega = \Omega - \{\omega\}, \omega \in \Omega - \Omega_0\}$ we obtain that $\bigcap \mathcal{R} = \Omega_0$, so that $\|\bigcap \mathcal{R}\| = \alpha$ and $\Pi(\bigcap \mathcal{R}) = \oslash_{\mathcal{T}}$ follows. But, $\|\Omega - \{\omega\}\| = \|\Omega\|$ for every $\omega \in \Omega$, hence, $\Pi(\Omega - \{\omega\}) = \mathbf{1}_{\mathcal{T}}$ and we obtain that $\Pi(\bigcap \mathcal{R}) = \oslash_{\mathcal{T}} \neq \bigwedge\{\Pi(A_\omega) : \omega \in \Omega - \Omega_0\} = \mathbf{1}_{\mathcal{T}}$, so that Π is not α-continuous from above on $\mathcal{P}(\Omega)$.

In general, however, $\mathcal{T}$-possibilistic measures defined on $\mathcal{P}(\Omega)$ need not be $\|\Omega\|$-continuous from above, as the following example demonstrates. Let $\Omega = [0,1]$, let $0 < a < 1$, let $\Pi([0,1]) \to T$ be such that $\Pi(A) = \oslash_{\mathcal{T}}$, if $A \subset [0,a]$ holds, $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ otherwise. Obviously, Π is a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$, as $\Pi(A \cup B) = \oslash_{\mathcal{T}}$ iff $A \cup B \subset [0,a]$, so that $A \subset [0,a]$ and $B \subset [0,a]$ holds, and $\Pi(A \cup B) = \mathbf{1}_{\mathcal{T}}$ iff there exists $a < x \leq 1$ such that $x \in A \cup B$; hence, $x \in A$ or $x \in B$, so that $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ or $\Pi(B) = \mathbf{1}_{\mathcal{T}}$. Let $\mathcal{R} = \{[0,a] \cup \{x\} : a < x \leq 1\}$. Then $\bigcap \mathcal{R} = [0,a]$; hence, $\Pi(\bigcap \mathcal{R}) = \oslash_{\mathcal{T}}$, but $\Pi([0,a] \cup \{x\}) = \mathbf{1}_{\mathcal{T}}$ for every $a < x \leq 1$. So, $\|[0,a] \cup \{x\}\| = \|\Omega\| = \aleph_1$ for each $a < x \leq 1$; hence, $\Pi([0,a] \cup \{x\}) = \mathbf{1}_{\mathcal{T}}$ and we obtain that $\| \bigcap \mathcal{R}\| = \|[0,a]\| = \aleph_1 = \|\Omega\|$, but $\|\Pi(\bigcap \mathcal{R}) = \oslash_{\mathcal{T}} \neq \bigwedge\{\Pi([0,a] \cup \{x\}) : a < x \leq 1\} = \mathbf{1}_{\mathcal{T}}$, so that Π is not $\|\Omega\|$-continuous from above on $\mathcal{P}(\Omega)$.

The notion of $\langle \alpha, \beta \rangle$-continuity from below does not offer so flexible tool as $\langle \alpha, \beta \rangle$-continuity from above as far as the classification of $\mathcal{T}$-valued possibilistic measures is concerned. As a matter of fact, each $\mathcal{T}$-possibilistic measure on the power-set $\mathcal{P}(\Omega)$ of the space Ω is $\langle \alpha, \beta \rangle$-continuous from below for each finite $\alpha = 1, 2, \ldots$ and each β. Indeed, for each finite $\mathcal{R} \subset \mathcal{P}(\Omega)$ the relation $\Pi(\bigcup \mathcal{R}) = \bigvee\{\Pi(A) : A \in \mathcal{R}\}$ is obviously valid, no matter which the values $\|A\|$ for $A \in \mathcal{R}$ may be. Hence, Π is $\langle \alpha, 0 \rangle$-continuous from below, so that it is (due to Lemma 4.1) $\langle \alpha, \beta \rangle$-continuous from below for each $\alpha = 1, 2, \ldots$ and for each β.

Let us consider, once more, the possibilistic measure Π defined on the power-set $\mathcal{P}(\Omega)$ of an infinite countable space Ω and ascribing the value $\Pi(A) = \oslash_{\mathcal{T}}$, if $A = \oslash$ or if A is finite, and ascribing $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ to infinite subsets of Ω. This measure is not $\langle \aleph_0, \alpha \rangle$-continuous from below no matter which α finite may be. Indeed, $\pi(\{\omega\}) = 0$ for every $\omega \in \Omega$ and the system $\{\{\omega\} : \omega \in \Omega\}$ is countable; however, $\mathbf{1}_{\mathcal{T}} = \Pi(\Omega) = \Pi(\bigcup\{\{\omega\} : \omega \in \Omega\}) \neq \bigvee\{\Pi(\{\omega\}) : \omega \in \Omega\} = \oslash_{\mathcal{T}}$. As a matter of fact, Π is $\langle \aleph_0, \aleph_0 \rangle$-continuous from below, as for every $\mathcal{R} \subset \mathcal{R}(\Omega)$ such that $\|A\| = \aleph_0$ for every $A \in \mathcal{R}$ we obtain that $\Pi(\bigcup \mathcal{R}) = \mathbf{1}_{\mathcal{T}} = \bigvee\{\Pi(A) : A \in \mathcal{R}\}$.

This construction may be easily shifted by one level higher. Let Ω be the space of the cardinality of continuum, i.e., $\|\Omega\| = c$ in symbols. Define $\pi : \mathcal{P}(\Omega) \to T$ in this way: $\Pi(A) = \oslash_{\mathcal{T}}$, if $\|A\| \leq \aleph_0$ holds, $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ otherwise, i.e., if $\|A\| = c$ (the hypothesis of continuum is accepted). As can be easily proved, Π is an $\aleph_0$-maxitive (σ-complete, in the standard terms) $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Indeed, let $\mathcal{R} \subset \mathcal{P}(\Omega)$ be such that $\|\mathcal{R}\| \leq \aleph_0$ holds. If $\|A\| \leq \aleph_0$ is the case for each $A \in \mathcal{R}$, then $\|\bigcup \mathcal{R}\| \leq \aleph_0$ also holds, so that $\Pi(A) = \oslash_{\mathcal{T}} = \Pi(\bigcup \mathcal{R})$ holds for each $A \in \mathcal{R}$, hence, the relation $\Pi(\bigcup \mathcal{R}) = \bigvee\{\Pi(A) : A \in \mathcal{R}\}$ results. If $\|A\| = c$ for some $A \in \mathcal{R}$, then the relation $\Pi(\bigcup \mathcal{R}) = \mathbf{1}_{\mathcal{T}} = \bigvee\{\Pi(A) : A \in \mathcal{R}\}$ holds again. As no restrictions have been imposed on the cardinalities $\|A\|$ of sets from $\mathcal{R}$, we may conclude that Π is $\langle \aleph_0, 0 \rangle$ (hence, also $\langle \aleph_0, \beta \rangle$ for every β) – continuous from below on $\mathcal{P}(\Omega)$.

However, Π is not $\langle c, \beta \rangle$-continuous from below on $\mathcal{P}(\Omega)$, no matter which $\beta \leq \aleph_0$ may be. As there exists the system of singletons, i.e., the system of cardinality c covering Ω, there exists, for each $0 < \beta \leq \aleph_0$, a system $\mathcal{R} \subset \mathcal{P}(\Omega)$ such that $\bigcup \mathcal{R} = \Omega$ and $\|A\| = \beta$ for each $A \in \mathcal{R}$, hence, $\Pi(\bigcup \mathcal{R}) = \Pi(\Omega) = \mathbf{1}_{\mathcal{T}} \neq \bigvee\{\Pi(A) : A \in \mathcal{R}\} = \oslash_{\mathcal{T}}$, so that π is not $\langle c, \beta \rangle$-continuous from below on $\mathcal{P}(\Omega)$. Obviously, Π is $\langle c, c \rangle$-continuous from below on $\mathcal{P}(\Omega)$.

5 LATTICE-VALUED POSSIBILISTIC MEASURES CONTINUOUS ON PARTICULAR SET SYSTEMS

Definition 5. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\mathcal{A}$ be a nonempty ample field of subsets of Ω, let Π be a $\mathcal{T}$-valued possibilistic measure on $\mathcal{A}$. Π is *continuous from above on* $\mathcal{R} \subset \mathcal{A}$, if $\Pi(\bigcap \mathcal{R}) = \bigwedge \{\Pi(A) : A \in \mathcal{R}\}$ holds, and Π is *continuous from below on* $\mathcal{R}$, if $\Pi(\bigcup \mathcal{R}) = \bigvee \{\Pi(A) : A \in \mathcal{R}\}$ is the case, here and below, $\bigcap \mathcal{R} = \bigcap_{A \in \mathcal{R}} A$ and $\bigcup \mathcal{R} = \bigcup_{A \in \mathcal{R}} A$.

Let us note that the sets $\bigcap \mathcal{R}$ and $\bigcup \mathcal{R}$ are always in $\mathcal{A}$, as $\mathcal{A}$ is an ample field, so that the values $\Pi(\bigcap \mathcal{R})$ and $\Pi(\bigcup \mathcal{R})$ are defined, but in general neither $\bigcap \mathcal{R} \in \mathcal{R}$ nor $\bigcup \mathcal{R} \in \mathcal{R}$ need be the case. Obviously, Π is continuous from above (or from below) on $\mathcal{R} \subset \mathcal{A}$, if $\bigcap \mathcal{R} \in \mathcal{R}$ or $\bigcup \mathcal{R} \in \mathcal{R}$ holds. E. g., let $\mathcal{A} = \mathcal{P}(\Omega)$, let $\Pi(A) = \mathbf{1}_{\mathcal{T}}$, if $A \subset \Omega, A \neq \emptyset$, let $\Pi(\emptyset) = \oslash_{\mathcal{T}}$. Then Π is continuous from above on every $\mathcal{R} \subset \mathcal{P}(\Omega)$ such that either $\emptyset \in \mathcal{R}$ or $\bigcap \mathcal{R} \neq \emptyset$, and Π is continuous from below on every $\mathcal{R} \subset \mathcal{P}(\Omega)$. Indeed, if $\mathcal{R} = \{\emptyset\}$, then $\Pi(\bigcup \mathcal{R}) = \Pi(\emptyset) = \oslash_{\mathcal{T}} = \bigvee \{\Pi(A) : A \in \mathcal{R}\}$, if there is $A \in \mathcal{R}, A \neq \emptyset$, then $\Pi(\bigcup \mathcal{R}) = \mathbf{1}_{\mathcal{T}} = \bigvee \{\Pi(A) : A \in \mathcal{R}\}$ holds as well.

If $\mathcal{R} \subset \mathcal{A}$ is a field of subsets of Ω, then Π is continuous from above as well as from below on $\mathcal{R}$. As a matter of fact, taking any $A \in \mathcal{R}$, the sets $\Omega - A, A \cap (\Omega - A) = \emptyset$ and $A \cup (\Omega - A) = \Omega$ are also in $\mathcal{R}$, so that $\bigcap \mathcal{R} = \emptyset \in \mathcal{R}$ and $\bigcup \mathcal{R} = \Omega \in \mathcal{R}$ holds and the simple fact from the last paragraph applies. In particular, Π is continuous from above as well as from below on both the extremum subsystems of $\mathcal{A}$, i.e., on $\{\emptyset, \Omega\}$ as well as on $\mathcal{A}$ itself, no matter which the $\mathcal{T}$-valued possibilistic measure Π on $\mathcal{A}$ may be.

Lemma 4. Let $\mathcal{T}, \Omega, \mathcal{A}$, and Π be as in Definition 5.1.

(i) Let $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{A}$ be such that $\bigcap \mathcal{R}_1 = \bigcap \mathcal{R}_2$ and Π is continuous from above on $\mathcal{R}_1$. Then Π is also continuous from above on $\mathcal{R}_2$.

(ii) Let $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{A}$ be such that $\bigcup \mathcal{R}_1 = \bigcup \mathcal{R}_2$ and Π is continuous from below on $\mathcal{R}_1$. Then Π is also continuous from below on $\mathcal{R}_2$.

Proof. Let the conditions of (i) hold. The inclusion $\mathcal{R}_1 \subset \mathcal{R}_2$ immediately implies the inequality

$$\bigwedge \{\Pi(A) : A \in \mathcal{R}_1\} \geq \bigwedge \{\Pi(B) : B \in \mathcal{R}_2\}. \tag{16}$$

Moreover, for each $B \in \mathcal{R}_2$ we obtain that $B \supset \bigcap \mathcal{R}_2 = \bigcap \mathcal{R}_1$ holds, so that $\Pi(B) \geq \Pi(\bigcap \mathcal{R}_1)$ holds for every $B \in \mathcal{R}_2$. Consequently

$$\Pi\left(\bigcap \mathcal{R}_1\right) = \Pi\left(\bigcap \mathcal{R}_2\right) = \bigwedge \{\Pi(A) : A \in \mathcal{R}_1\} \geq \bigwedge \{\Pi(B) : B \in \mathcal{R}_2\} \geq \Pi\left(\bigcap \mathcal{R}_1\right) \tag{17}$$

and the equality

$$\Pi\left(\bigcap \mathcal{R}_2\right) = \bigwedge \{\Pi(B) : B \in \mathcal{R}_2\} \tag{18}$$

follow, so that Π is continuous from above on $\mathcal{R}_2$.

Let the conditions of (ii) hold. For each $B \in \mathcal{R}_2$ the inclusion $\bigcup \mathcal{R}_2 \supset B$ and, consequently, the inequalities

$$\Pi\left(\bigcup \mathcal{R}_2\right) \geq \Pi(B), \Pi\left(\bigcup \mathcal{R}_2\right) \geq \bigvee\{\Pi(B) : B \in \mathcal{R}_2\} \tag{19}$$

are valid. So, we obtain that the relation

$$\begin{aligned} \bigvee\{\Pi(A) : A \in \mathcal{R}_1\} &= \Pi\left(\bigcup \mathcal{R}_1\right) = \Pi\left(\bigcup \mathcal{R}_2\right) \geq \bigvee\{\Pi(B) : B \in \mathcal{R}_2\} \geq \\ &\geq \bigvee\{\Pi(A) : A \in \mathcal{R}_1\} \end{aligned} \tag{20}$$

holds, so that the equality

$$\Pi\left(\bigcup \mathcal{R}_2\right) = \bigvee\{\Pi(B) : B \in \mathcal{R}_2\} \tag{21}$$

follows. Hence, Π is continuous from below on $\mathcal{R}_2$ and the assertion is proved. □

Besides the continuity from above and from below *on* a system $\mathcal{R} \subset \mathcal{A}$ of subsets of Ω let us introduce also the notion of continuity of a possibilistic measure Π from above and from below *in* a particular set $A \in \mathcal{A}$.

Definition 6. Let $\mathcal{T}, \Omega, \mathcal{A}$, and Π be as in Definition 5.1, let $A \in \mathcal{A}$. The possibilistic measure Π defined on $\mathcal{A}$ is called *continuous from above in* A, if it is continuous from above on each $\mathcal{R} \subset \mathcal{A}$ such that $\bigcap \mathcal{R} = A$, i.e., if $\Pi\left(\bigcap \mathcal{R}\right) = \bigwedge\{\Pi(B) : B \in \mathcal{R}\}$ holds for each $\mathcal{R} \subset \mathcal{A}$ such that $\bigcap \mathcal{R} = A$. Dually, Π *is continuous from below in* A, if it is continuous from below on each $\mathcal{R} \subset \mathcal{A}$ such that $\bigcup \mathcal{R} = A$, i.e., if $\Pi\left(\bigcup \mathcal{R}\right) = \Pi(A)$ holds for each $\mathcal{R} \subset \mathcal{A}$ such that $\bigcup \mathcal{R} = A$.

Abbreviated, $\Pi \downarrow A$ or $\Pi \uparrow A$ denotes that the $\mathcal{T}$-possibilistic measure Π is continuous from above (or from below) in $A \in \mathcal{A}$.

Theorem 2. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice such that, for each $\emptyset \neq S \subset T$ and each $t \in T$, the relation

$$\bigwedge\{s \wedge t : s \in S\} = \left(\bigwedge S\right) \wedge t \tag{22}$$

holds (let us recall that the inequality $\bigwedge\{s \wedge t : s \in S\} \geq (\bigwedge S) \wedge t$ holds in general). Let $\Omega, \mathcal{A}$, and Π be as in Definition 5.1, let Π be continuous from above in $A \in \mathcal{A}$. Then Π is continuous from above in every $B \supset A, B \in \mathcal{A}$.

Proof. Let $B \in \mathcal{A}, B \supset A$, let $\mathcal{R} \subset \mathcal{A}$ be such that $\bigcup \mathcal{R} = B$ so that, for each $C \in \mathcal{R}$, the inclusion $C \supset B \supset A$ holds. Let $\mathcal{R}_1 = \{C - (B - A) : C \in \mathcal{R}\}$. Then

$$\begin{aligned} \bigcap \mathcal{R}_1 &= \bigcap\{C - (B - A) : C \in \mathcal{R}\} = \bigcap\{C \cap (B - A)^C : C \in \mathcal{R}\} = \\ &= (B - A)^C \cap \bigcap\{C : C \in \mathcal{R}\} = (B^C \cup A) \cap B = A \cap B = A. \end{aligned} \tag{23}$$

As Π is continuous from above in A, we obtain

$$\Pi(A) = \Pi\left(\bigcap \mathcal{R}_1\right) = \bigwedge\{\Pi(C) : C \in \mathcal{R}_1\}. \tag{24}$$

Hence, for every $\mathcal{R} \subset \mathcal{A}$ such that $\bigcap \mathcal{R} = B$ there exists $\mathcal{R}_1 \subset \mathcal{A}$ such that $\bigcap \mathcal{R}_1 = A$ and for each $C \in \mathcal{R}$ we have $C = C_1 \cup (B - A)$ for some $C_1 \in \mathcal{R}_1$. Consequently,

$$\Pi(C) = \Pi(C_1) \vee \Pi(B - A) \tag{25}$$

for such a C_1, inversely, for each $C_1 \in \mathcal{R}_1$ the set $C_1 \cup (B - A)$ is in $\mathcal{R}$. So,

$$\begin{aligned} & \bigwedge\{\Pi(C) : C \in \mathcal{R}\} = \bigwedge\{\Pi(C_1) \vee \Pi(B - A) : C_1 \in \mathcal{R}_1\} = \\ = \; & \left(\bigwedge\{\Pi(C_1) : C_1 \in \mathcal{R}_1\}\right) \vee \Pi(B - A) = \Pi(A) \vee \Pi(B - A) = \\ = \; & \Pi(A \cup (B - A)) = \Pi(B) \end{aligned} \tag{26}$$

due to the assumptions imposed on $\mathcal{T}$. So, the relation

$$\Pi(B) = \Pi\left(\bigcap \mathcal{R}\right) = \bigwedge\{\Pi(C) : C \in \mathcal{R}\} \tag{27}$$

is valid, so that Π is continuous from above in B. □

Theorem 3. Let $\mathcal{T}, \Omega, \mathcal{A}$, and Π be as in Definition 5.1, let Π be continuous from above in $A \in \mathcal{A}$, let $B \in \mathcal{A}, B \subset A$ be such that $\Pi(A) = \Pi(B)$. Then Π is continuous from above in B.

Proof. Let $\mathcal{R}_1 \subset \mathcal{A}$ be such that $\bigcap \mathcal{R}_1 = B$, so that $C \supset B$ and $\Pi(C) \geq \Pi(B) = \Pi(A)$ holds for every $C \in \mathcal{R}_1$. Consequently, the inequality

$$\bigwedge\{\Pi(C) : C \in \mathcal{R}_1\} \geq \Pi(B) = \Pi(A) \tag{28}$$

also holds. Let $\mathcal{R} = \{C \cup (A - B) : C \in \mathcal{R}_1\}$, then

$$\bigcap \mathcal{R} = \bigcap\{C \cup (A - B) : C \in \mathcal{R}_1\} = \bigcap\{C : C \in \mathcal{R}_1\} \cup (A - B) = B \cup (A - B) = A. \tag{29}$$

As Π is continuous from above in A, we obtain that

$$\Pi\left(\bigcap \mathcal{R}\right) = \bigwedge\{\Pi(C \cup (A - B)) : C \in \mathcal{R}_1\} = \bigwedge\{\Pi(C) \vee \Pi(A - B) : C \in \mathcal{R}_1\} = \Pi(A). \tag{30}$$

For each $C \in \mathcal{R}$, the inequality $\Pi(C) \vee \Pi(A - B) \geq \Pi(C)$ is obvious, so that the inequality

$$\bigwedge\{\Pi(C) : C \in \mathcal{R}_1\} \leq \bigwedge\{\Pi(C \cup (A - B)) : C \in \mathcal{R}_1\} = \Pi(A) = \Pi(B) \tag{31}$$

easily follows. Combining (28) and (31) we obtain that $\bigwedge\{\Pi(C) : C \in \mathcal{R}_1\} = \Pi(B)$, so that Π is continuous from above in B. □

There exists a trivial example of complete lattice-valued possibilistic measure defined on whole $\mathcal{P}(\Omega)$ which is continuous from above on each $\mathcal{R} \subset \mathcal{P}(\Omega)$ and in each $A \subset \Omega$ – take simply $\mathcal{T} = \langle \mathcal{P}(\Omega), \subset \rangle$ and define $\Pi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ as the identity Π_{id} on $\mathcal{P}(\Omega)$. Indeed, the demand $\Pi(\bigcap \mathcal{R}) = \bigwedge\{\Pi(A) : A \in \mathcal{R}\}$

easily reduces to the identity $\bigcap \mathcal{R} = \bigcap \mathcal{R}$ for each $\mathcal{R} \subset \mathcal{P}(\Omega)$. Slightly modifying this trivial construction we can obtain, again for $\mathcal{T} = \langle \mathcal{P}(\Omega), \subset \rangle$, a complete $\mathcal{T}$-valued possibilistic measure which is continuous from above in each $A \subset \Omega$ containing a given $A_0 \subset \Omega, A_0 \neq \Omega$, as its proper subset, i.e., $A_0 \subset A$, $A_0 \neq A$ is supposed to hold. Let $\Pi_{A_0}(A) = \emptyset$, if $A \subset A_0$, let $\Pi_{A_0}(A) = A \cup A_0$ otherwise, so that, in particular, $\Pi_{A_0}(A) = A$ for every $A \supset A_0, A \neq A_0$. Then $\Pi_{A_0}(\emptyset) = \emptyset$ and $\Pi_{A_0}(\Omega) = \Omega$ obviously holds and for any $\mathcal{R} \subset \mathcal{P}(\Omega)$ such that $\bigcup \mathcal{R} \subset A_0$ is the case we obtain that $A \subset A_0$ holds for each $A \in \mathcal{R}$, so that $\Pi_{A_0}(A) = \emptyset$ for every $A \in \mathcal{R}$ and the relation $\Pi(\bigcup \mathcal{R}) = \emptyset = \bigcup \{\Pi(A) : A \in \mathcal{R}\}$ is valid. If $(\bigcup \mathcal{R}) - A_0 \neq \emptyset$, then there is $A_1 \in \mathcal{R}$ such that $A_1 - A_0 \neq \emptyset$ holds; hence,

$$\begin{aligned}
\Pi_{A_0}\left(\bigcup \mathcal{R}\right) &= A_0 \cup \left(\bigcup \mathcal{R}\right) = A_0 \cup \bigcup \{A : A \in \mathcal{R}_1, A - A_0 \neq \emptyset\} = \\
&= A_0 \cup \bigcup \{\Pi_{A_0}(A) : A \in \mathcal{R}, A - A_0 \neq \emptyset\} = \\
&= \bigcup \{\Pi_{A_0}(A) : A \in \mathcal{R}, A - A_0 \neq \emptyset\} = \\
&= \bigcup \{\Pi_{A_0}(A) : A \in \mathcal{R}\}, \qquad (32)
\end{aligned}$$

as $\Pi_{A_0}(A) \supset A_0$ holds at least for $A = A_1 \in \mathcal{R}$ and $\Pi_{A_0}(A) = \emptyset$ for every $A \in \mathcal{R}, A \subset A_0$, i.e., for every $A \in \mathcal{R}, A - A_0 = \emptyset$. Hence, Π_{A_0} is a complete $\langle \mathcal{P}(\Omega), \subset \rangle$-valued possibilistic measure on $\mathcal{P}(\Omega)$.

Let $A_1 \subset \Omega$ be such that $A_0 \subset A_1, A_0 \neq A_1$ holds, let $\mathcal{R} \subset \mathcal{P}(\Omega)$ be such that $\bigcap \mathcal{R} = A_1$ is the case. Then there exists $\omega_1 \in A_1 - A_0$, hence, for each $A \in \mathcal{R}$ the inclusion $A \supset A_0 \cup \{\omega_1\}$ is valid, so that $\Pi_{A_0}(A) = A$ and $\bigcap \{\Pi_{A_0}(A) : A \in \mathcal{R}\} = A_1 = \Pi_{A_0}(A_1) = \Pi_{A_0}(\bigcap \mathcal{R})$. So, Π_{A_0} is continuous from above in any $A \subset \Omega$ containing A_0 as its *proper* subset. However, Π_{A_0} is not continuous from above in A_0 itself. Indeed, take $\mathcal{R} = \{A_0 \cup \{\omega_1\} : \omega_1 \in \Omega - A_0\}$, then $\bigcap \mathcal{R} = A_0$ and $\Pi_{A_0}(\bigcap \mathcal{R}) = \emptyset$, but $\Pi_{A_0}(A_0 \cup \{\omega_1\}) = A_0 \cup \{\omega_1\}$ for every $\omega_1 \in \Omega - A_0$, so that $\bigcap \{\Pi_{A_0}(A_0 \cup \{\omega_1\}) : \omega_1 \in \Omega - A_0\} = \bigcap \{A_0 \cup \{\omega_1\} = \omega_1 \in \Omega - A_0\} = A_0$. If $A_0 \neq \emptyset$, this value differs from $\Pi(\bigcap \mathcal{R}) = \emptyset$.

A simple result valid in the particular case of real-valued possibilistic measures defined by possibilistic distributions on the universe Ω is perhaps worth being introduced explicitly.

Theorem 4. Let Ω be a nonempty set, let $\pi : \Omega \rightarrow [0, 1]$ be a real-valued possibilistic distribution on Ω, so that $\bigvee_{\omega \in \Omega} \pi(\omega) = 1$, let $\Pi : \mathcal{P}(\Omega) \rightarrow [0, 1]$ be the possibilistic measure defined when setting $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$ for every $A \subset \Omega$. Then Π is continuous from above in $A \subset \Omega$ iff there is at most one $\omega \in \Omega$ such that $\pi(\omega) > \Pi(A)$ holds.

Proof. Let there be $\omega_1, \omega_2 \in \Omega$ such that $\pi(\omega_i) > \Pi(A)$ is the case for both $i = 1, 2$. Consequently, $\omega_i \in \Omega - A$ holds for both $i = 1, 2$, so that, taking $\mathcal{R} = \{A \cup \{\omega_1\}, A \cup \{\omega_2\}\} \subset \mathcal{P}(\Omega)$, we obtain that $\bigcap \mathcal{R} = A$. However, the inequality

$$\begin{aligned}\Pi(A \cup \{\omega_1\}) \wedge \Pi(A \cup \{\omega_2\}) &= (\Pi(A) \vee \pi(\omega_1)) \wedge (\Pi(A) \vee \pi(\omega_2)) = \\ &= \pi(\omega_1) \wedge \pi(\omega_2) > \Pi(A) \end{aligned} \tag{33}$$

follows immediately, so that Π is not continuous from above in A.

If there is no $\omega \in \Omega$ such that $\Pi(A) < \pi(\omega)$ holds, then $\Pi(A) = 1$, hence, $\Pi(B) = 1$ for each $B \in \mathcal{R}$ such that $\bigcap \mathcal{R} = A$, so that Π is continuous from above in A (let us recall also the more general result from above concerning the sets with $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ in the case of lattice-valued $\mathcal{T}$-possibilistic measures).

If there is just one $\omega_0 \in \Omega$ such that $\pi(\omega_0) > \Pi(A)$ holds, we conclude easily that $\Pi(A) < 1$ and $\pi(\omega_0) = 1$ must be the case. Indeed, if $\pi(\omega_0) < 1$ would hold, then there must exist another $\omega_1 \in \Omega - A$ such that $\pi(\omega_1) > \pi(\omega_0)$ is valid in order to meet the demand that $\bigvee_{\omega \in \Omega} \pi(\omega) = 1$. As $A \cup \{\omega_0\} \neq A$, there must be, in every $\mathcal{R} \subset \mathcal{P}(\Omega)$ such that $\bigcap \mathcal{R} = A$, either the set A itself, or another set $B \supset A$ such that $\Pi(B) = \Pi(A)$ is the case. Then the identity

$$\Pi\left(\bigcap \mathcal{R}\right) = \Pi(A) = \bigwedge \{\Pi(B) : B \in \mathcal{R}\} \tag{34}$$

is valid, so that Π is continuous from above in A. The assertion is proved. □

Acknowledgement

This work was sponsored by the grant No. IAA100300503 of the GA AS CR. This work was partly supported by the Institutional Research Plan AV0Z10300504 "Computer Science for the Information Society: Models, Algorithms, Applications".

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