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GENERALIZATION OF ONE-SIDED CONCEPT LATTICES

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Abstract. We provide a generalization of one-sided (crisp-fuzzy) concept lattices, based on Galois connections. Our approach allows analysis of object-attribute models with different structures for truth values of attributes. Moreover, we prove that this method of creating one-sided concept lattices is the most general one, i.e., with respect to the set of admissible formal contexts, it produces all Galois connections between power sets and the products of complete lattices. Some possible applications of this approach are also included.

Keywords: One-sided concept lattices, Galois connection, closure operator, concept data analysis

Mathematics Subject Classification 2010: 06A15, 06B99, 68T30

1 INTRODUCTION

Formal Concept Analysis (FCA, [9]) is a theory of data analysis for identification of conceptual structures among data sets. FCA has been found useful in concept data analysis, knowledge discovery, text mining, information retrieval, as well as in other areas from machine learning and artificial intelligence. FCA known as the theory of concept lattices is based on the notion of formal context, which is represented by the binary relation between the set of objects and the set of attributes. From a formal context, one can construct (objects, attributes) pairs known as the formal concepts. Then hierarchical structure of formal concepts, ordered by the generalization and specialization among the concepts, forms a concept lattice, which is a type of concept hierarchy where each node represents a subset of objects (extent) with the corresponding set of attributes (intent).

In classic approach to FCA the authors provide crisp case, where object-attribute model is based on the binary relation (object has/has-not the attribute). In practice there are natural examples of object-attribute models for which relationship between objects and attributes are represented by fuzzy relations. Therefore, several attempts to fuzzify FCA have been proposed.

We mention an approach of Bělohlávek [1, 2, 3] based on logical framework of complete residuated lattices, work of Georgescu and Popescu to extend this framework to non-commutative logic [10, 11, 12], an approach of Krajči [16], Popescu [20], an approach of Medina, Ojeda-Aciego, Ruiz-Calviño [17, 18], and also an approach of one of the authors [19] generalizing all approaches based on Galois connections. A survey and comparison of some existing approaches to fuzzy concept lattices is presented in [4].

In fuzzy FCA a special role is played by one-sided concept lattices, where usually objects are considered as a crisp subset and attributes obtain fuzzy values. In this case interpretation of object clusters is straightforward as in classical FCA, instead of fuzzy approaches with fuzzy subsets of objects, where interpretability often becomes problematic. From existing one-sided approaches we mention papers by Krajči [15], Yahia and Jaoua [5], Jaoua and Elloumi [14], or by Zhang, Ma, Fan [22]. Practical applications of one-sided concept lattices can be found in the monograph by Carpineto and Romano [8], in the above mentioned papers [5, 14, 15], or also in [6, 7].

All recently known approaches allow only one type of structure for truth degrees. However, it is reasonable to consider object-attribute models with different truth value structures for their attributes. The main aim of this paper is to generalize one-sided concept lattices in order to satisfy such idea. It means that we are able to combine different truth value structures for attributes, e.g., qualitative attributes with possible values 0 and 1, quantitative attributes from some real-valued interval, ordinal attributes, etc.

Theory of concept lattices is closely related to Galois connections and closure systems, hence in the preliminary section we give brief overview of these notions. In the following section we provide generalization of one-sided concept lattices, some basic properties of this new approach, and incremental algorithm for creation of one-sided concept lattice from given formal context. There is also an illustrative example of the creating the one-sided concept lattice. In Section 4 we prove that the presented approach is the most general one, i.e., it produces all possible Galois connections between power sets and products of complete lattices. Another interesting feature of our approach is the possibility for merging the various formal contexts. In previous approaches to one-sided concept lattices such combination was possible only in the case where all attributes obtain values from one truth value structure. These theoretical considerations are also described in this section. At the end of this section we provide an example of generalized one-sided formal context, which combines several overlapping subcontexts with different types of attributes and their truth value structures.

2 PRELIMINARIES

In this section we briefly describe algebraic framework for FCA and give a basic overview of the Galois connections needed for our purposes.

First we provide the basic notions of FCA as described by Ganter and Wille in [9]. Let (B, A, I) be a formal context, i.e., $B, A \neq \emptyset$ and $I \subseteq B \times A$. A pair of mappings ${}^{\uparrow}\colon 2^B \to 2^A$ and ${}^{\downarrow}\colon 2^A \to 2^B$ is defined as follows:

$$X^{\uparrow} = \{y \in A : (x, y) \in I \text{ for all } x \in X\},$$

$$Y^{\downarrow} = \{x \in B : (x, y) \in I \text{ for all } y \in Y\}.$$

Note that 2^S denotes the power set of the set S, i.e., the set of all subsets of S. Consequently a set

$$\mathcal{B}(B, A, I) = \{(X, Y) : X \subseteq B, Y \subseteq A, X^{\uparrow} = Y, X = Y^{\downarrow}\}$$

is given.

$\mathcal{B}(B, A, I)$ forms a complete lattice called concept lattice. This fact yields from the properties of pair of mappings $({}^{\uparrow}, {}^{\downarrow})$, which forms so-called Galois connection.

Next we provide necessary details regarding Galois connections and related topics. We assume that the reader is familiar with the basic notions of theory of partially ordered sets.

Definition 1. Let $(P, \leq)$ and $(Q, \leq)$ be an ordered sets and let

$$\varphi\colon P \to Q \quad \text{and} \quad \psi\colon Q \to P$$

be maps between these ordered sets. Such a pair (φ, ψ) of mappings is called a *Galois connection* between the ordered sets if:

a) $p_1 \leq p_2$ implies $\varphi(p_1) \geq \varphi(p_2)$,
b) $q_1 \leq q_2$ implies $\psi(q_1) \geq \psi(q_2)$,
c) $p \leq \psi(\varphi(p))$ and $q \leq \varphi(\psi(q))$.

The two maps are also called *dually adjoint* to each other. We note that

$$\varphi = \varphi \circ \psi \circ \varphi \quad \text{and} \quad \psi = \psi \circ \varphi \circ \psi$$

and that the conditions a), b) and c) are equivalent to the following one:

d) $p \leq \psi(q)$ if and only if $\varphi(p) \geq q$.

Theory of concept lattices is based on the framework of algebraic structures known as complete lattices. For lattice theory the terminology and the notation as in Grätzer [13] will be used. Recall that by complete lattice we understand a partially ordered set $(L, \leq)$, such that for each subset $X \subseteq L$ there exist supremum (the least upper bound) of X, and infimum (the greatest lower bound) of X. In the sequel we will use the symbol $\bigvee$ for supremum and $\bigwedge$ for infimum, respectively. Each complete lattice L possesses the greatest element 1_L and the least element 0_L, satisfying $0_L \leq x \leq 1_L$ for all $x \in L$. The class of all complete lattices will be denoted by CL.

We will use the following well known fact (see [9], [21]):

Lemma 1. A map $\varphi\colon L \to M$ between complete lattices has a dual adjoint if and only if

$$\varphi\left(\bigvee_{i \in I} x_i\right) = \bigwedge_{i \in I} \varphi(x_i)$$

holds for any subset $\{x_i : i \in I\}$ of L.

In this case the dual adjoint ψ is uniquely determined by

$$\psi(y) = \bigvee \{x \in L : \varphi(x) \geq y\}.$$

Galois connections between complete lattices are closely related to the notion of *closure operator* and *closure system*. Let L be a complete lattice. By a closure operator in L we understand a mapping $c_L\colon L \to L$ satisfying:

a) $x \leq c_L(x)$ for all $x \in L$,

b) $c_L(x_1) \leq c_L(x_2)$ for $x_1 \leq x_2$,

c) $c_L(c_L(x)) = c_L(x)$ for all $x \in L$, (i.e., c_L is idempotent).

A subset X of a complete lattice L is called a closure system in L if X is closed under arbitrary meets. We note that this condition guarantees that $(X, \leq)$ is a complete lattice, in which the infima are the same as in L, but the suprema in X may not coincide with those from L. For a closure operator c_L in L, the set $X(c_L)$ of all fixed points of c_L (i.e., $X(c_L) = \{x \in L : c_L(x) = x\}$) is a closure system in L. Conversely, for a closure system X in L, a mapping $c_L^X\colon L \to L$ defined by $c_L^X(x) = \bigwedge\{u \in X : x \leq u\}$ is a closure operator in L. Moreover, these correspondences are inverses of each other, i.e., $X(c_L^X) = X$ for each closure system X in L and $c_L^{X(c_L)} = c_L$ for each closure operator c_L in L.

Next, we recall the relationship between the closure operators induced by the Galois connections. Two ordered sets P, Q are called dually isomorphic, if there is an antitone (order reversing) bijective mapping $f\colon P \to Q$ such that f^{-1} is also antitone. Let $L, M \in \mathsf{CL}$ and (φ, ψ) be a Galois connection between L and M. Then the mapping $\varphi \circ \psi \colon L \to L$ is a closure operator in L; similarly, $\psi \circ \varphi \colon M \to M$ is a closure operator in M. Moreover the corresponding closure systems are dually isomorphic.

Conversely, suppose that X_1 and X_2 are closure systems in L, M, respectively, and $f \colon X_1 \to X_2$ is a dual isomorphism between complete lattice $(X_1, \leq)$ and complete lattice $(X_2, \leq)$. Then a pair $(c_{L_1} \circ f, c_{L_2} \circ f^{-1})$, where c_{L_1}, c_{L_2} are closure operators corresponding to X_1 and to X_2, forms a Galois connection between L and M. Hence, any Galois connection between complete lattices induces dually isomorphic closure systems on these lattices and vice versa.

The properties of Galois connections allow us to construct complete lattices which are in our special interest. Formally, let (φ, ψ) be a Galois connection between complete lattices L and M. Denote by $\mathcal{G}_{\varphi,\psi}$ a subset of $L \times M$ consisting of all pairs (x, y) with $\varphi(x) = y$ and $\psi(y) = x$. Define a partial order on $\mathcal{G}_{\varphi,\psi}$ as follows:

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{if} \quad x_1 \leq x_2 \text{ iff } y_1 \geq y_2 \text{ (due to condition d) of Definition 1).}$$

Proposition 1. Let (φ, ψ) be a Galois connection between complete lattices L and M. Then $(\mathcal{G}_{\varphi,\psi}, \leq)$ forms a complete lattice, where

$$\bigwedge_{i \in I} (x_i, y_i) = \left(\bigwedge_{i \in I} x_i, \varphi\left(\psi\left(\bigvee_{i \in I} y_i\right)\right)\right), \quad \bigvee_{i \in I} (x_i, y_i) = \left(\psi\left(\varphi\left(\bigvee_{i \in I} x_i\right)\right), \bigwedge_{i \in I} y_i\right)$$

for each family $(x_i, y_i)_{i \in I}$ of elements from $\mathcal{G}_{\varphi,\psi}$.

Let us remark that lattices of the form $\mathcal{G}_{\varphi,\psi}$ are considered as fuzzy concept lattices or Galois lattices. If one of the lattices L or M is equal to the power set 2^B of some non-empty set B, we will refer to $\mathcal{G}_{\varphi,\psi}$ as to one-sided fuzzy concept lattices.

3 GENERALIZATION OF ONE-SIDED FUZZY CONCEPT LATTICES

In this section we provide a generalization of one-sided fuzzy concept lattices, which was independently described by Krajči [15] and by Yahia and Jaoua [5]. In this case the input consists of an L-context (B, A, R) where L is a complete lattice and $R \colon B \times A \to L$ is a binary L-fuzzy relation. Again, there is a pair of mappings $\beta \colon 2^B \to L^A$ and $\alpha \colon L^A \to 2^B$, which forms Galois connection and is defined as follows:

$$\beta(X)(a) = \bigwedge_{b \in X} R(b, a),$$

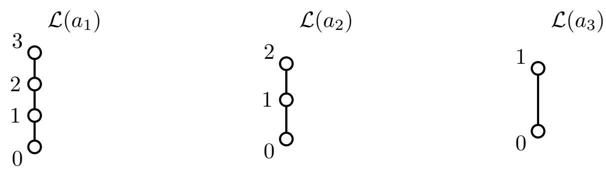
$$\alpha(g) = \{b \in B : \text{ for each } a \in A,\ g(a) \leq R(b, a)\}.$$

In order to obtain a generalization of one-sided approach we have to introduce the notion of formal one-sided context which only little differs from that commonly used.

Definition 2. A 4-tuple $c = (B, A, \mathcal{L}, R)$ is said to be a *generalized one-sided formal context* if the following conditions are fulfilled:

1. B is a non-empty set of objects and A is a non-empty set of attributes.
2. $\mathcal{L}\colon A \to \mathsf{CL}$ is a mapping from the set of attributes to the class of all complete lattices. Hence, for any attribute a, $\mathcal{L}(a)$ denotes a structure of truth values for attribute a.
3. R is generalized incidence relation, i.e., $R(b, a) \in \mathcal{L}(a)$ for all $b \in B$ and $a \in A$. Thus, $R(b, a)$ represents a degree from the structure $\mathcal{L}(a)$ in which the element $b \in B$ has the attribute a.

Example 1. Let $B = \{a, b, c, d\}$ be the set of object and $A = \{a_1, a_2, a_3\}$ be the set of attributes. Next, we put $\mathcal{L}(a_1) = \mathbf{4}$, $\mathcal{L}(a_2) = \mathbf{3}$ and $\mathcal{L}(a_3) = \mathbf{2}$, where $\mathbf{4}$, $\mathbf{3}$ and $\mathbf{2}$ denotes four, three and two element chain, respectively.



Generalized incidence relation R is captured in the following table:

R	a_1	a_2	a_3
a	2	0	1
b	3	1	1
c	1	1	0
d	3	2	1

Now we provide a basic result about generalized one-sided concept lattices.

Definition 3. Let $(B, A, \mathcal{L}, R)$ be a generalized one-sided formal context. We define a pair of mappings $\uparrow\colon 2^B \to \prod_{a \in A} \mathcal{L}(a)$ and $\downarrow\colon \prod_{a \in A} \mathcal{L}(a) \to 2^B$ as follows:

$$\uparrow(X)\,(a) = \bigwedge_{b \in X} R(b, a) \tag{1}$$

$$\downarrow(g) = \{b \in B : \text{ for each } a \in A,\ g(a) \le R(b, a)\}. \tag{2}$$

Theorem 1. Let $(B, A, \mathcal{L}, R)$ be a generalized one-sided formal context. Then a pair $(\uparrow, \downarrow)$ forms a Galois connection between 2^B and $\prod_{a \in A} \mathcal{L}(a)$.

Proof. Let X_i, $i \in I$ be a family of subsets of the set B. Then for any $a \in A$ we obtain:

$$\uparrow\left(\bigcup_{i \in I} X_i\right)(a) = \bigwedge_{b \in \bigcup_{i \in I} X_i} R(b, a) = \bigwedge_{i \in I} \left(\bigwedge_{b \in X_i} R(b, a)\right) = \bigwedge_{i \in I} \uparrow(X_i)\,(a).$$

The middle equality follows from the fact that in $\mathcal{L}(a)$ the set of all lower bounds of the set $\{R(b,a) : b \in \bigcup_{i\in I} X_i\}$ equals to the set of all lower bounds of the set $\{\bigwedge_{b\in X_i} R(b,a) : i \in I\}$. Hence $\uparrow\left(\bigcup_{i\in I} X_i\right) = \bigwedge_{i\in I} \uparrow(X_i)$ and according to Lemma 1, the mapping $\uparrow$ has an unique dual adjoint ψ.

Due to Lemma 1, $\psi(g) = \bigcup\{X : X \subseteq B, \uparrow(X) \geq g\}$. The mapping $\uparrow$ is antitone, thus $b \in \psi(g)$ if and only if $\uparrow(\{b\}) \geq g$. Since $\uparrow(\{b\}) \geq g$ if and only if for each $a \in A$, $\uparrow(\{b\})(a) = R(b,a) \geq g(a)$, we obtain

$$b \in \psi(g) \text{ iff } b \in \downarrow(g) = \{b' \in B : \text{ for each } a \in A,\ g(a) \leq R(b',a)\}.$$

This yields $\psi = \uparrow$, thus $(\uparrow, \downarrow)$ forms a Galois connection between 2^B and $\prod_{a\in A} \mathcal{L}(a)$. □

Now we are able to define generalized one-sided concept lattice. For formal context $(B, A, \mathcal{L}, R)$ denote by $\mathcal{C}(B, A, \mathcal{L}, R)$ the set of all pairs (X, g), where $X \subseteq B$, $g \in \prod_{a\in A} \mathcal{L}(a)$, satisfying

$$\uparrow(X) = g \quad \text{and} \quad \downarrow(g) = X.$$

Set X is usually referred to as *extent* and g as *intent* of the concept (X, g).

Further we define partial order on $\mathcal{C}(B, A, \mathcal{L}, R)$ as follows:

$$(X_1, g_1) \leq (X_2, g_2) \text{ iff } X_1 \subseteq X_2 \text{ iff } g_1 \geq g_2.$$

According to Proposition 1 we obtain the following theorem.

Theorem 2. Let $(B, A, \mathcal{L}, R)$ be a generalized one-sided formal context. Then $\mathcal{C}(B, A, \mathcal{L}, R)$ with the partial order defined above forms a complete lattice, where

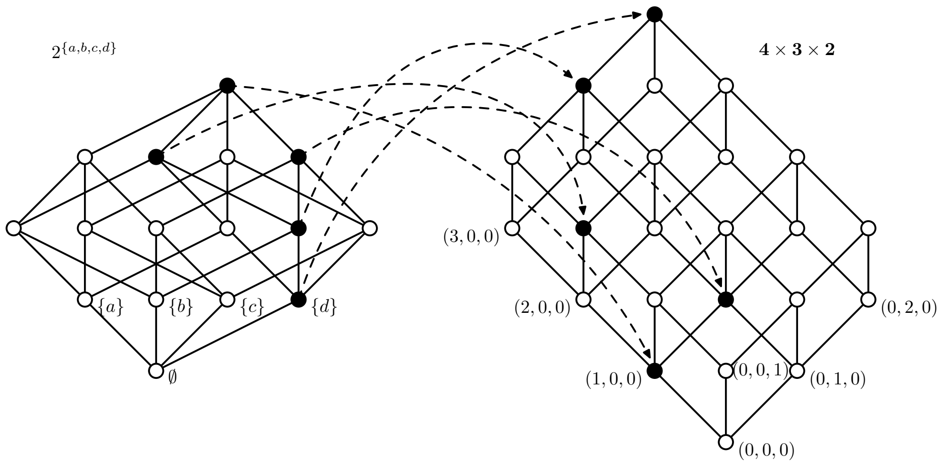
$$\bigwedge_{i\in I}(X_i, g_i) = \left(\bigcap_{i\in I} X_i, \uparrow\downarrow\left(\bigvee_{i\in I} g_i\right)\right), \quad \bigvee_{i\in I}(X_i, g_i) = \left(\downarrow\uparrow\left(\bigcup_{i\in I} X_i\right), \bigwedge_{i\in I} g_i\right)$$

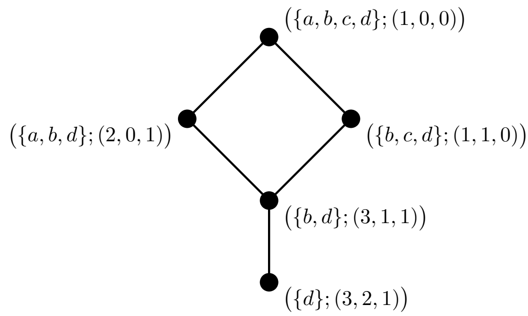
for each family $(X_i, g_i)_{i\in I}$ of elements from $\mathcal{C}(B, A, \mathcal{L}, R)$.

We illustrate the process of creating one-sided concept lattice from given context in the following example.

Example 2. Consider the generalized one-sided formal context from Example 1. Applying the definitions (1) and (2) we obtain Galois connection $(\uparrow, \downarrow)$ between $2^{\{a,b,c,d\}}$ and $\mathcal{L}(a_1) \times \mathcal{L}(a_2) \times \mathcal{L}(a_3) = \mathbf{4} \times \mathbf{3} \times \mathbf{2}$. Note that for more legibility we only indicate the corresponding dual isomorphism of closure systems. This obviously uniquely determines the corresponding Galois connection.

From this Galois connection we obtain the following generalized one-sided concept lattice $\mathcal{C}(B, A, \mathcal{L}, R)$.





We enclose this section providing an incremental algorithm for creation of one-sided concept lattice.

Let $(B, A, \mathcal{L}, R)$ be a generalized one-sided formal context. For $b \in B$ put $R(b)$ an element of $\prod_{a\in A} \mathcal{L}(a)$ such that $R(b)(a) = R(b,a)$, i.e., $R(b)$ represents b-th row in data table R. Let 1_L denote the greatest element of $L = \prod_{a\in A} \mathcal{L}(a)$, i.e., $1_L(a) = 1_{\mathcal{L}(a)}$ for all $a \in A$.

Algorithm

`Input:` `generalized formal context` $(B, A, \mathcal{L}, R)$
`begin`
  `create lattice` $L := \prod_{a\in A} \mathcal{L}(a)$
  $C := \{1_L\}$, $C \subseteq L$ `is the set of all intents`
  `while`$(B \neq \emptyset)$
  $\{$
    `choose` $b \in B$
    $C^* := C$
    `for each` $c \in C^*$

```
        C := C ∪ {c ∧ R(b)}
      B := B ∖ {b}
    }
    for each c ∈ C
      𝒞(B, A, ℒ, R) := 𝒞(B, A, ℒ, R) ∪ {(↓(c), c)}
end
Output:  set of all concepts 𝒞(B, A, ℒ, R)
```

Correctness of the algorithm yields from the following facts. Evidently, C is the smallest closure system in L containing $\{R(b) : b \in B\}$. Since $R(b) = \uparrow(\{b\})$, we obtain $C \subseteq \uparrow\left(2^B\right)$. Conversely, if $g = \uparrow(X) \in \uparrow\left(2^B\right)$, then $g = \bigwedge_{b\in X} \uparrow(\{b\}) = \bigwedge_{b\in X} R(b) \in C$. Hence $C = \uparrow\left(2^B\right)$.

Let us remark that step for creation of the lattice $L := \prod_{a\in A} \mathcal{L}(a)$ can be done in various ways and it is up to programmer. For example, it is not necessary to store all elements of $\prod_{a\in A} \mathcal{L}(a)$, but it is sufficient to store only particular lattices $\mathcal{L}(a)$, since lattice operations in L are calculated component-wise.

4 POSSIBLE APPLICATION OF GENERALIZED ONE-SIDED CONCEPT LATTICES

In this section we point out one of the possible applications of generalized one-sided concept lattices, i.e., merging of various subcontexts. First we show that generalized one-sided concept lattices represent the most general approach of creating one-sided concept lattices. Particularly, we prove that any Galois connection between power set and the product of complete lattices can be obtained using generalized incident relation. This fact allows to represent various one-sided concept lattices as formal contexts, and further to involve these contexts to the creation of another one-sided concept lattices. The main result of this section is based on the following two simple assertions.

Lemma 2. Let B be a non-empty set and L be a complete lattice. Then any two Galois connections (ϕ_1, ψ_1), (ϕ_2, ψ_2) between 2^B and L are equal if and only if $\phi_1(\{b\}) = \phi_2(\{b\})$ for all $b \in B$.

Proof. Obviously, $(\phi_1, \psi_1) = (\phi_2, \psi_2)$ if and only if $\phi_1 = \phi_2$. Since

$$\phi(X) = \phi\left(\bigcup_{b\in X}\{b\}\right) = \bigwedge_{b\in X} \phi(\{b\})$$

for any Galois connection $(\phi, \psi) \in \mathbf{Gal}(2^B, L)$, we obtain $\phi_1 = \phi_2$ if and only if $\phi_1(\{b\}) = \phi_2(\{b\})$ for all $b \in B$. □

Omitting the fact that the system of all complete lattices with Galois connections and composition of functions does not form a category, this lemma can be briefly stated as "power sets are free objects". In particular, power sets 2^B together with the set of generators B have the Universal Mapping Property, i.e., there is a function $\iota\colon B \to 2^B$, $\iota(b) = \{b\}$ for all $b \in B$, and for any $L \in \mathsf{CL}$ and any mapping $f\colon B \to L$ there is a unique Galois connection (ϕ, ψ) between $2^B$and L such that $f = \iota \circ \phi$, all as indicated in the following diagram.

$$\begin{tikzcd}
2^B \arrow[r, "\phi"] & L \\
B \arrow[u, "\iota"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Corollary 1. Let (ϕ, ψ) be a Galois connection between 2^B and $\prod_{a\in A} \mathcal{L}(a)$, where $B, A \neq \emptyset$ and $\mathcal{L}(a) \in \mathsf{CL}$ for all $a \in A$. Then there exists a generalized one-sided formal context $c = (B, A, \mathcal{L}, R)$ such that $\phi = \uparrow_c$ and $\psi = \downarrow_c$.

Proof. For any $b \in B$ and $a \in A$ we put $R(b, a) = \phi(\{b\})(a)$. Since $\uparrow_c(\{b\})(a) = R(b, a) = \phi(\{b\})(a)$ for all $a \in A$, we obtain $\uparrow_c(\{b\}) = \phi(\{b\})$ for all $b \in B$, which yields $\uparrow_c = \phi$. The dual adjoint is unique, hence $\downarrow_c = \psi$. □

This result shows that essentially there is no other way how to define Galois connection from the given one-sided formal context. Further, we provide another possible application of Corollary 1 and our new approach of generalized one-sided concept lattices.

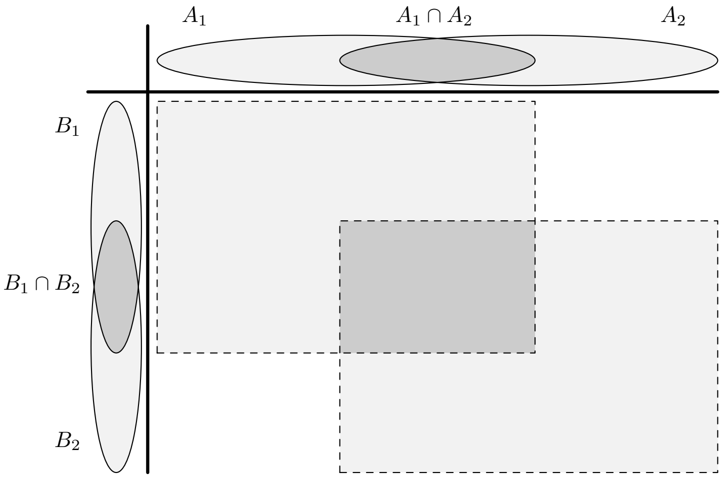
4.1 Combination of Different One-Sided Approaches

In the analysis of the object-attribute models it is sometimes necessary to deal with the problem of overlapping input entries. We can describe this problem as follows. Let B be a non-empty set of objects and A be a non-empty set of attributes. Further, let $B_1, B_2 \subseteq B$ represent two subsets of objects and $A_1, A_2 \subseteq A$ represent two subsets of attributes. Assume that B_1 has been already investigated against set A_1 of attributes. Similarly, B_2 has been analyzed against A_2. Each of the previous analyses could be performed using other one-sided approaches (e.g., [14, 16, 22]). The next figure illustrates this situation.

Formally, generalized subcontexts $c_1 = (B_1, A_1, \mathcal{L}_1, R_1)$ and $c_2 = (B_2, A_2, \mathcal{L}_2, R_2)$ are given, such that $\mathcal{L}_1(a) = \mathcal{L}_2(a)$ for all $a \in A_1 \cap A_2$, i.e., the lattices corresponding to overlapping attributes are equal.

Our aim is to find a generalized context $c = (B_1 \cup B_2, A_1 \cup A_2, \mathcal{L}, R)$, which includes both subcontexts c_1 and c_2. For $a \in A_1 \cup A_2$ we put $\mathcal{L}(a)$ as follows:

$$\mathcal{L}(a) = \begin{cases} \mathcal{L}_1(a); & \text{if } a \in A_1, \\ \mathcal{L}_2(a); & \text{otherwise.} \end{cases}$$



Further, we define generalized incident relation R as follows:

$$R(b,a) = \begin{cases} R_1(b,a); \text{ if } b \in B_1 \text{ and } a \in A_1 \setminus A_2, \text{ or } b \in B_1 \setminus B_2 \text{ and } a \in A_1, \\ R_2(b,a); \text{ if } b \in B_2 \text{ and } a \in A_2 \setminus A_1, \text{ or } b \in B_2 \setminus B_1 \text{ and } a \in A_2, \\ R_1(b,a) \wedge R_2(b,a); \text{ if } b \in B_1 \cap B_2 \text{ and } a \in A_1 \cap A_2, \\ R_e(b,a); \text{ otherwise.} \end{cases}$$

In first two cases the value of $R(b,a)$ is obvious. The last case represents the situation when no information is available from subcontexts c_1 and c_2. In this case $R_e(b,a)$ is an arbitrary value from $\mathcal{L}(a)$ and can be obtained in different ways, e.g., it is an experimental piece of data from measurement or some expert choice.

If $b \in B_1 \cap B_2$ and $a \in A_1 \cap A_2$, then information from both subcontexts is available. In general, one can define $R(b,a) = R_1(b,a) \otimes R_2(b,a)$, where $\otimes\colon \mathcal{L}(a) \times \mathcal{L}(a) \to \mathcal{L}(a)$ denotes an arbitrary binary operation on $\mathcal{L}(a)$. In order to count in values from both subcontexts meaningfully, we proposed to use the meet operation in the lattice $\mathcal{L}(a)$. In the next paragraph we try to explain our choice.

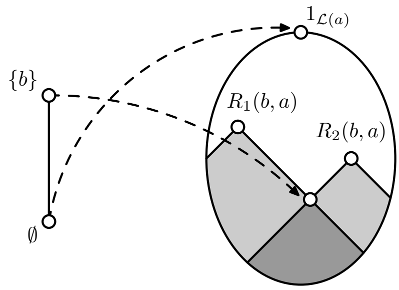
For any object b, any attribute a and $R(b,a)$ from $\mathcal{L}(a)$ one can consider the following simple formal context $c_{b,a} = (\{b\}, \{a\}, \mathcal{L}, R)$ consisting of one-element object and attribute set respectively. From Definition 3, we obtain Galois connection $(\uparrow_{c_{b,a}}, \downarrow_{c_{b,a}})$ between the lattice $2^{\{b\}}$ (isomorphic to the two element chain) and the lattice $\mathcal{L}(a)$, such that

$$\uparrow_{c_{b,a}}(\{b\}) = R(b,a) \uparrow_{c_{b,a}}(\emptyset) = 1_{\mathcal{L}(a)}$$

and

$$\downarrow_{c_{b,a}}(x) = \{b\} \text{ for all } x \leq R(b,a), \downarrow_{c_{b,a}}(x) = \emptyset \text{ otherwise.}$$

Hence, the value of $R(b,a)$ can be considered as some kind of threshold for an element x of $\mathcal{L}(a)$ to be mapped onto $\{b\}$. Since $R(b,a)$ uniquely determines Galois connection $(\uparrow_{c_{b,a}}, \downarrow_{c_{b,a}})$, any generalized incident relation R of one-sided formal context $c = (B, A, \mathcal{L}, R)$ can be viewed as a system of Galois connections $\left(\uparrow_{c_{b,a}}, \downarrow_{c_{b,a}}\right)_{(b,a)\in B\times A}$. It was proved in [19] that resulting Galois connection $(\uparrow_c, \downarrow_c)$ from Definition 3 can be equivalently defined by the system $\left(\uparrow_{c_{b,a}}, \downarrow_{c_{b,a}}\right)_{(b,a)\in B\times A}$ of Galois connections, such that each particular Galois connection $(\uparrow_{c_{b,a}}, \downarrow_{c_{b,a}})$ is involved to the creation of final Galois connection $(\uparrow_c, \downarrow_c)$.



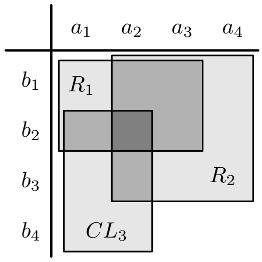
From this point of view, if we want to map an element $x \in \mathcal{L}(a)$ to $\{b\}$ if and only if x was mapped to $\{b\}$ by both Galois connections derived from subcontexts c_1 and c_2 respectively, it is reasonable to put the threshold $R(b,a) = R_1(b,a) \wedge R_2(b,a)$. In this case $x \leq R(b,a)$ if and only if $x \leq R_1(b,a)$ and $x \leq R_2(b,a)$ as it is requested for threshold $R(b,a)$.

4.2 Example

Now we provide an illustrative example of previous considerations and possible applications of our approach. Assume that we have set B of four objects, say b_1, b_2, b_3, b_4, which we want to analyse and find some interesting clusters (subsets) of them. Also we have some previous experiences from analysis of some subsets of B against different subsets of attributes a_1, a_2, a_3, a_4. For truth values of attributes we assume the following complete lattices. Attributes a_1 and a_3 are binary, thus $\mathcal{L}(a_1) = \mathcal{L}(a_3) = \mathbf{2}$. Attribute a_2 is quantitative with values from real unit interval, i.e. $\mathcal{L}(a_2) = [0,1]$. Finally, the values of a_4 are represented by modular non-distributive lattice M_3 (also known as a diamond), which consists of the smallest element o, the greatest element i, and three incomparable elements a, b, c satisfying $o < a < i$, $o < b < i$, $o < c < i$. Our aim is to create an appropriate object-attribute model from given data using approach based on the generalized one-sided formal context.

The following figure shows our partial knowledge about the context of example.

We assume that some previous investigation of objects from $B_1 = \{b_1, b_2\}$ and attributes $A_1 = \{a_1, a_2, a_3\}$ yields generalized incident relation R_1. Similarly, there is a generalized formal subcontext $(B_2, A_2, \mathcal{L}, R_2)$ with $B_2 = \{b_1, b_2, b_3\}$ and



$A_2 = \{a_2, a_3, a_4\}$. Finally, the subset $B_3 = \{b_2, b_3, b_4\}$ was investigated against the subset $A_3 = \{a_1, a_2\}$ of attributes. From this investigation, only concept lattice CL_3 and hence Galois connection between 2^{B_3} and $\mathcal{L}(a_1) \times \mathcal{L}(a_2)$ is available. In this case $CL_3 = \big\{ (\emptyset, (1, 1.0))\,; (\{b_3\}, (1, 0.3))\,; (\{b_2, b_4\}, (0, 0.8))\,; (\{b_2, b_3, b_4\}, (0, 0.3)) \big\}$. According to Corollary 1 there is generalized incident relation R_3, such that $CL_3 = \mathcal{C}\,(B_3, A_3, \mathcal{L}, R_3)$. Relations R_1, R_2, R_3 are summarized in the following tables.

R_1	a_1	a_2	a_3
b_1	0	0.7	1
b_2	1	0.45	1

R_2	a_2	a_3	a_4
b_1	0.55	0	a
b_2	0.6	1	o
b_3	0.2	1	b

Now we can create generalized incident relation R for generalized one-sided context $(B, A, \mathcal{L}, R)$. We will use the procedure from Section 4.1.

Since there is no information about the relationship of object b_4 and attributes a_3, a_4, we put some "expert" knowledge for these values.

The resulting generalized one-sided concept lattice is depicted in the next figure.

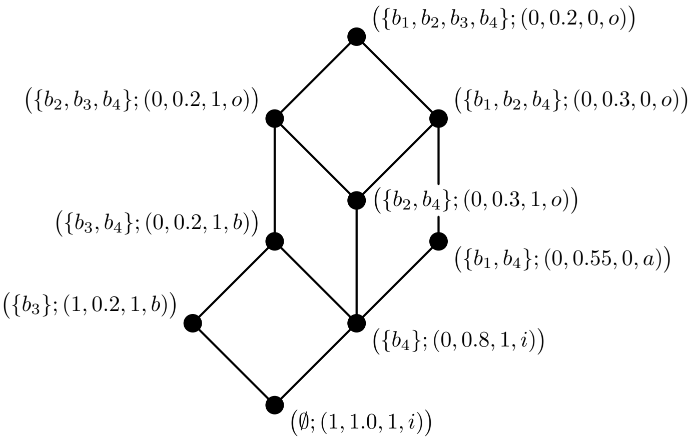
5 CONCLUSIONS

In this paper we have introduced an approach for generalization of one-sided concept lattices based on Galois connection, which is applicable in concept data analysis of object-attribute models. The basic idea of this generalization is based on the usage of different truth value structures on the side of attributes. It was proved that this method produces all crisp-fuzzy Galois connections, hence our method includes all known approaches of creating the one-sided concept lattices based on the Galois connections.

Main application benefit of this method is merging of various subcontexts, which provide the possibility to analyze object-attribute models with partial and (in some

R_3	a_1	a_2
b_2	0	0.8
b_3	1	0.3
b_4	0	0.8

R	a_1	a_2	a_3	a_4
b_1	0	0.55	0	a
b_2	0	0.3	1	o
b_3	1	0.2	1	b
b_4	0	0.8	1	i



cases) overlapping knowledge about the objects and their relationships to the attributes. The presented approach is applicable in all domains where all other one-sided approaches were used, e.g., concept data analysis, knowledge discovery, text mining, and information retrieval.

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